Accounting for Framing-Effects - an informational approach to intensionality in the Bolker-Jeffrey decision model

Sacha Bourgeois-Gironde, Raphaël Giraud

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Sacha Bourgeois-Gironde
Institut Jean-Nicod
CNRS-ENS-EHESS

Raphaël Giraud
CRESE
University of Franche-Comté†

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Empirically observed cognitive biases (Kahneman, Slovic, and Tversky, 1982; Kahneman and Tversky, 2000) raise a challenge to the more formal and normative approach to rationality. Logicians, when facing these cases, can be led to reflect on some basic principles of theirs. Among these biases, framing effects (Tversky and Kahneman, 1981) occur when different descriptions of the same decision problem give rise to divergent decisions. However, a widely accepted, though seldom explicitly formulated[1], principle of decision theoretic models states that preferences should not be affected by a variation in the description of the problem (Arrow, 1982; Tversky and Kahneman, 2000). We shall refer to this principle, which is an invariance principle, as the *principle of extensionality*. In logic, indeed, the principle of extensionality requires that two formulas having always the same truth value for any truth assignments be substitutable *salva veritate* in another formula. The invariance principle is but an extension of this principle to decision contexts. Framing effects, then, exemplify a violation of the principle of extensionality prevailing in decision theory.

Different interpretations have been proposed for framing effects under their various guises, mainly from a purely psychological point of view. Because they have long been discarded as irrational and because formal models of decision theory have for a long time only cared about rational behavior, there are very few formal models of framing effects. Obviously the introduction of framing effects into formal models entails a recasting of

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[1]See Cohen and Jaffray (1980), Lipman (1999), Rubinstein (2000) and Le Menestrel and Van Wassenhove (2001) for some explicit formulations.

decision theory. In the most famous of these formal models, Prospect Theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992), framing effects motivate the introduction of reference dependence and different behavior in the domains of gains and losses[2]. Until recently this model remained the only one for framing effects. Some alternative models currently emerge (Gold and List, 2004; Giraud, 2004). Our intention in this paper is to contribute both to the philosophical literature and the formal literature on framing effects.

Our more specific intention is to try to understand in what sense framing effects present violations of the principle of extensionality, i.e. what kind of violation they precisely are. Our first aim, then, is to point to the source of extensionality in decision theory. Frederic Schick (Schick, 1992) has held that extensionality is imposed on us by basic axioms of a certain decision theory, namely Bolker-Jeffrey's logic of decision (Bolker, 1966, 1967; Jeffrey, 1965): "I think the message is this. The usual logic does indeed impose extensionality on us. It might be said that it is part of its purpose, part of what it is for. If we want more freedom here - if we want to free ourselves to set different utilities on propositions we think co-reportive - we need a less constraining logic" (Schick, 1992, p. 35). This comes just after an attempted derivation of extensionality from a principle of conditional utilities Schick puts at the basis of his decision theory. We will criticize Schick's understanding and treatment of his own point but we retain the main lesson: in order to account for violations of extensionality as a widespread feature in decision making some modification in the structure of decision theory is required. We will also keep in the decision theoretical framework proposed by Bolker and Jeffrey that Schick uses without making an explicit mention of it.

Our choice will accordingly be to revise a theory of decision for which preferences bear on propositions. This choice is congenial to an approach, like ours, whose aim is to connect back together logical and philosophical analyses of intensionality, on the one hand, and derivation of framing effects from axioms of decision theory, on the other hand; hence, for our purpose, the relevance of the Bolker-Jeffrey framework for a logic of decision.

In the first section, we shall discuss why intensionality may be normatively grounded. We will briefly discuss several extant philosophical analyses of how information processing, as opposed to perception of logical equivalence, can enter into decision making procedures or explain features of contexts of decision. The purpose of this discussion is to understand in traditional philosophical terms and in cognitive terms what authors have meant when they said framing effects were violations of extensionality, that is instantiated a kind of intensional phenomenon. Our next purpose, in section 2, is then to see how,

[2]Specifically, risk aversion in the domain of gains and risk propensity in the domain of losses.

however, intensionality is precluded in the Bolker-Jeffrey framework. We will investigate the source of extensionality in this framework and see why it cannot simply accommodate framing effects. As we shall show, Bolker-Jeffrey decision theory implicitly relies on the entailment of indifference from logical equivalence, calling logically equivalent two sentences that express the same proposition (i.e. they are true in the same set of possible worlds). Our bypassing strategy will then be, in the third section, to shift attention from logical equivalence to a notion of co-reportiveness in the original Bolker-Jeffrey framework, assuming that two propositions are co-reportive if they have the same factual interpretation in the possible worlds deemed relevant by the decision-maker. We will then show how this allows, first, to define a stronger notion of equivalence of propositions, namely information equivalence, and, second, to see how this relation is connected to intensionality.

1 Intensionality and Why it may be Normatively Grounded

What does it mean to characterize framing effects as a breakdown of extensionality? How can violations of extensionality, as displayed by framing effects, be intuitively and formally related to extant views of intensionality in philosophical logic?

Schick suggests a notion of intensionality for framing effects: subjects who are aware of the extensional equivalence[3] of two propositions may still prefer one to the other: "I want to arrange for a logic of action that allows for people's understandings. This means that the logic of action I want has to be intensional. More strictly, it means that its theory of value has to be intensional, that the theory may not fault someone's setting different utilities on propositions he thinks co-reportive"(Schick, 1992, p. 33). Under Schick's conception, framing-effects seem to imply that even though the extensional equivalence of two propositions A and B is subjectively realized, they may not be substitutable in choice or valuation contexts. However, it is hard to make a pronouncement about the degree of acknowledgment of co-extensionality by subjects in the absence of empirical data on individual perception of equivalence between options in a decision problem (see Frisch (1993) for a treatment of this problem). Moreover, Schick's suggestion calls for a formalization which we shall attempt to provide.

To make Schick's statement more precise, a different approach, one defended for instance by Craig McKenzie at the principle of his experimental studies (Sher and McKenzie, 2005), would have it that even though co-reportiveness[4] is subjectively perceived

[3]We use this phrase as synonymous to co-reportiveness.

[4]McKenzie uses the term "logical equivalence" instead of "co-reportiveness". He uses this term in the following sense "a pair of statements is logically equivalent if each member of the pair necessarily entails the other". The two concepts are not the same. However, they are intimately connected: if two propositions

and admitted between A and B, A and B are not information equivalent for the subjects as they may bear distinct connotations and license different inferences. In McKenzie's terms: " Suppose (...) that speakers, choosing between uttering 'A' and uttering 'B', are more likely to utter 'A' when some background condition C holds than when C fails. In that case, a listener who hears a speaker say 'A' can safely infer a higher probability of C being true than if the speaker had said 'B', that is, $p(C \mid \text{speaker says '}A\text{'}) > p(C \mid \text{speaker says '}B\text{'})$. If knowledge about the background condition C is relevant to the choice at hand, then the speaker's utterance of the two logically equivalent statements A and B may with impunity lead to different decisions" (Sher and McKenzie, 2005) . So it is clear that subjects may recognize co-reportiveness ("logical equivalence" in McKenzie's terms) and at the same time informational discrepancies between two sentences. To use a rather trivial illustration of this point, one can, with no difficulty, recognize that "half empty" and "half full" are co-extensional attributes in the sense that they refer to one single current state of a glass of wine. But different inferences may be drawn by a listener from the use of one attribute instead of the other. One may infer from the description of the glass as "half empty" that it was formerly full and from its description as "half full" that it was previously empty. This is the kind of underlying inferences from the choice of certain descriptions in decision problems that may explain why co-reportive sentences may be treated as informatively distinct.

Violations of extensionality in decision making are cognitively grounded — and normatively defensible – if they involve overlooking either logical equivalence in the sense of McKenzie or co-reportiveness in the sense of Schick but not information equivalence. What is interesting in Schick and McKenzie's respective characterizations of framing effects is that they focus on two different facets of the problems, both connected to the classical notion of intensionality. Schick points to the plausible cognitive state of the agent when he chooses A over B: even though he is aware of the co-reportiveness of A and B he values one rather than the other because he prefers the way the same and self-identical option is framed with A than the way it is with B. He values A as a frame more than B as a frame, notwithstanding his cognitive lucidity about what A and B commonly denote, namely the same object. As to McKenzie, he simply makes sense of this cognitive state through his notion of pragmatic inference to the most plausible information in the background. He, so to say, connects a cognitive state to a cognitive competence. He collapses framing effects to inference about the best information in order to make one's mind in a decision problem.

are co-reportive, they will be logically equivalent. To take Frege's example, a sentence about the morning star is logically equivalent, in McKenzie's sense, to the same sentence where the phrase "the morning star" is replaced by "the evening star", because these two phrases are co-reportive, and so are the sentences that contain them.

In the traditional Fregean context of birth of intensional logic, a lack of apprehension of co-referentiality (i.e. co-reportiveness) between two terms can lead to the non-application of a principle of logic: the substitution *salva veritate* of these co-referential terms in a sentential context. In order to account for this phenomenon, Frege introduces the notion of sense: "If we now replace one word of the sentence by another having the same reference, but a different sense, this can have no bearing upon the reference of the sentence. (...) What else but the truth value could be found, that belongs quite generally to every sentence if the reference of its component is relevant, and remains unchanged by substitutions of the kind in question?"(Frege, 1892). For a given pair of coextensional terms, the subject is not always aware that substitution of those terms preserves sentential truth value. Variation of sense without variation of reference does not affect the truth-value of a sentence but this is precisely what the agent does not see in this particular case. This is why his or her sensitivity to senses, rather than to reference, affects his or her apprehension of the two logically equivalent sentences as such. Sensitivity to senses is another phrase for what we are interested in in the present context, to wit intensionality. Cognitive states diverge from objective data. But it is striking to note the following asymmetry: the non perception of objective states, in Frege's approach, prevents that substitution of co-referential expressions be respected, while, in framing effects as we now understand them, perception of co-referentiality does not entail that co-referential sentences will be treated as such in decision contexts.

Both Schick and McKenzie, by their respective special focus on some aspects of the problem of the intensionality of framing effects, shed light on this alleged asymmetry.

What is interesting with framing effects, as Schick understands them, is that they help raise some fundamental points about a distinction between co-reportiveness and informational equivalence, and their role in decision contexts. Co-reportiveness is an "objective" datum while information equivalence is a subjective one. What Frege seems to imply in the quoted passage, is that lucidity about co-referenciality normatively entails a lack of sensitivity to senses in assessing the *truth* of a corresponding sentence. But this does not have to be the case in assessing the *utility* of the fact extensionally described by this same sentence. Schick then points to the cognitive dissociation between value and truth. Intensionality of frames, in Schick's mouth, is that frames, which are assigned the same reference, may not be assigned the same value. Truth, then, is not the only ground upon which to assess utility, and this is the starting point Schick proposes for the elaboration of an intensional decision theoretic framework, like the one we're going to sketch.

According to McKenzie, two sentences are information equivalent if no choice-relevant information can be inferred from one of them and not from the other. His notion of intensionality – his condition for framing effects to occur – is, in turn, the non systematic entailment of information equivalence from the acceptance of logical equivalence between options. He allows us to keep framing effects fully rational, in the sense that the agent is cognitively justified to make his choice, given that he infers the presence of a relevant piece of information in the context. Intensionality, thus, fulfils its classical function which is to preserve the agent's cognitive consistency or more generally his rationality. In Frege's perspective, senses (or intensions) are introduced in order to explain why certain inferences or certain substitutions are not performed in some contexts. With framing effects, the notion of "lack of information equivalence" allows us to maintain a cognitive consistency between the ability to draw particular inferences and express corresponding preferences. Because we do not necessarily perceive as information equivalent some descriptions or some sentences referring back to some possible state of affairs, we are justified in preferring one to the other. That choices are "intensional" means that they cognitively depend on this particular ability of ours to process information in certain regular ways. We, then, avoid what would be a more profound discrepancy between processing of information and application of logic, on one side, and choice, on the other side, one that would be given by the picture of an agent processing information in one way and choosing in another way.

So we are content to say that the partial links we emphasized between logical and informational equivalences capture a good deal of what could be meant when applying this notion of intensionality to framing effects. We accept that no phenomenon of framing effects can be connected with irrationality under McKenzie's view, if there is always a background information C that a listener can infer from the choice of some frame or description by a speaker. We do not consider that framing effects are the symptom of a more irreducible form of cognitive inconsistency between information-processing on the one hand and choice and decision-making on the other. In the present article we will thus try to capture in decision-theoretical terms the notion that Schick and McKenzie have proposed, that is to export the notion of intensionality from the logico-philosophical tradition inaugurated by Frege to the analysis of decision contexts.

2 Extensionality in the Bolker-Jeffrey Model

In this section, we want to assess Schick's claim that extensionality is forced onto us by basic axioms of (Bolker and Jeffrey's) decision theory. Frederic Schick is one of the few philosophers trying to argue for the very controversial view that allowing for understandings, as he puts it, i.e. allowing for different perceptions of the objects of choice, may be the key to understanding framing effects. He has tried to show that this approach of framing effects is compatible with a non-trivial view of framing effects, by trying to demonstrate that it suffices to relax some axioms of a decision theory akin to the Bolker and Jeffrey decision theory, while retaining others, in order to account for framing effects, or, in other words, to account for intensionality. His first step was to show that some primitive assumptions on utility and belief imply extensionality. Accordingly, one should weaken these assumptions in order to construct a sound decision theory allowing for understandings.

Unfortunately, it seems to us that Schick's attempt to derive extensionality from axioms of preference is fundamentally flawed, because he never provides a formal statement of what he is supposed to prove, so that, according to some natural definition of it, his proof is circular, arguably using what it is supposed to show, whereas, according to some other definition, he proves something else, a weaker result than the one he thinks he has proved. We think, however, that it is of interest to flesh out what Schick has actually proved, as it provides some nice formalizations of an extension of Frege's notion of extensionality to decision contexts.

In this section, we shall first review the basic features of the Bolker and Jeffrey decision model, then examine the result of Schick's attempts at deriving extensionality in this model, and finally single out what is arguably the true source of extensionality in the model.

2.1 The Bolker-Jeffrey Decision Model

The most distinctive feature of Bolker and Jeffrey's decision model is that in this model, preferences are interpreted as bearing on propositions. Before being more specific on the precise formalization of this notion in the model, let us first outline the meaning and the potential shortcomings of this interpretation. Let φ and ψ be two formulas (well-formed sentences) of some propositional language Φ and let $\succsim$ be a binary relation expressing preference. The phrase "$\varphi \succsim \psi$" must be understood as expressing the fact that the decision maker prefers that the proposition expressed by φ be true rather than the proposition expressed by ψ. Therefore, this statement of preference may be understood as saying that the decision maker prefers the world to be in a certain state described by φ to its being in the state described by ψ. The preference expressed is thus more the expression of a wish than the expression of some preference for a specific action.

The fact that, in this model, objects of choice are propositions is a strength, as it allows dealing with the fact that, in many cases, decisions to be made appear in the guise of a description of how the world will be if such and such decision is made. However, it involves collapsing the notion of an action with the state of affairs that will arise as a consequence of that action. To use a vocabulary that may sound more familiar to decision theorists, this theory confounds acts and outcomes. This is unsurprising as one of the motives for developing this theory was the idea that the usual definition of an act as some function from an independently defined set of states of the world to a set of outcomes was unsatisfactory, as the state of the world that obtains may well depend on the act chosen. However, at an abstract level where acts are not defined as functions, it is a little bit puzzling for a theory of decision only to deal with the values the decision maker attributes to the states in which the world may be, but not really with the choice of an action that may lead to a preferred state. To be sure, many actions can lead to the same state, so that this is clearly an issue. To be completely fair, however, one may note that it is always possible to consider the action chosen as part of the description of the world, so that preferring some state of the world involves preferring the actions that are undertaken in that state. However, this could be done more explicitly.

Let us now turn to the specificities of the model. We have said that in the model preferences bear on propositions. To be precise, they bear in fact on non-null elements of a complete, atom free Boolean algebra[5]. The interpretation of elements of a Boolean algebra as propositions requires some manipulation on which we shall come back later on. Let us only remark that extensionality heavily relies on this identification.

To be specific, let $< \mathscr{A}, \wedge, \vee, \neg >$ be the Boolean algebra of interest. Let $\top$ and $\bot$ be its extremal elements and let $\mathscr{A}' := \mathscr{A} \setminus \{\bot\}$. We consider a binary relation $\succsim$ defined on $\mathscr{A}'$ representing preferences. This preference relation is assumed to satisfy the following axioms.

BJ 1. *$\succsim$ is complete and transitive.*

BJ 2. *For all $a, b \in \mathscr{A}'$, if $a \wedge b = \bot$, then*

$$a \succ b \Rightarrow a \succ a \vee b \succ b$$

[5]A Boolean algebra is a tuple $< \mathscr{A}, \wedge, \vee, \neg >$ where $\mathscr{A}$ is a non-empty set, $\wedge$, $\vee$ are binary operations and $\neg$ a unary operation satisfying:

(i) $a \vee a = a$ and $a \wedge a = a$;

(ii) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;

(iv) $a \vee (b \wedge c) = (a \vee b) \wedge (b \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

(v) $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$;

(vi) There exist elements $\top$ and $\bot$ such that , $\top \wedge a = a$ and $\bot \vee a = a$;

(vii) $a \wedge \neg a = \bot$ and $a \vee \neg a = \top$.

The canonical example of a Boolean algebra is the set of all subsets of a given set X together with the natural set operations of union, intersection and complementation.

A Boolean algebra is complete if the binary operations can be performed on an arbitrary number of elements (i.e., if $(a_i)_{i \in I}$ is a family of elements of $\mathscr{A}$, then $\bigvee_{i \in I} a_i \in \mathscr{A}$ and $\bigwedge_{i \in I} a_i \in \mathscr{A}$).

A Boolean algebra is naturally endowed with a partial order defined by $a \leq b$ if and only if $a \vee b = b$. It is atom free if for any $a \in \mathscr{A}$, there exists a', a'' such that $a', a'' \leq a$, $a', a'' \neq a$ and $a = a' \vee a''$.

and

$$a \sim b \Rightarrow a \sim a \vee b \sim b.$$

We say that two elements a, b are disjoint if $a \wedge b = \bot$.

BJ 3. *For all $a, b, c \in \mathscr{A}'$, pairwise disjoint, if $a \sim b \not\sim c$ and $a \vee c \sim b \vee c$ then, for all d disjoint from a and b, $a \vee d \sim b \vee d$.*

BJ 4. *For any monotone sequence (a_n) in $\mathscr{A}'$, converging to a (i.e. $a = \bigvee a_n$ or $a = \bigwedge a_n$), if $b \succ a \succ c$, then there exists $N \in \mathbb{N}$ such that $b \succ a_n \succ c$ for all $n \geq N$.*

We now give the version of Bolker's theorem that will be of use in the sequel of this paper.

Theorem 1. *Under axioms BJ 1 to BJ 4, there exist a function $U : \mathscr{A}' \to \mathbb{R}$ and a probability measure $P : \mathscr{A}' \to [0, 1]$ such that*

1. $U(a) \geq U(b) \Leftrightarrow a \succsim b$.

2. *For all $a, b \in \mathscr{A}'$, $U(a) = U(a \wedge b)P(b \mid a) + U(a \wedge \neg b)P(\neg b \mid a)$,*

with $P(b \mid a) := \dfrac{P(a \wedge b)}{P(a)}$.

As can be seen in condition 2, this theorems axiomatizes a version of expected utility, that shall be of further use in the sequel.

2.2 Schick's Derivations of Extensionality

2.2.1 The First Derivation

For the first derivation, Schick introduces the concept of conditional utility. For any formulas h and k, $u(h, k)$ is the conditional utility of h given k: in Schick's words, it is "the agents's degree of disposition to want h if, in addition to all else he believed, he now believed k." Unconditional utility of proposition h is denoted by $u(h)$.

Schick mentions as being "a familiar principle of conditional utilities" the following relationship between conditional and unconditional utility:

$$\forall h, k, u(h, k) = u(h \wedge k). \tag{1}$$

What Schick aims at showing is relatively unclear from a formal point of view but what he claims to have proved is that, given this principle, if the agent believes that h

and n are co-reportive in the words of Schick, a relation he denotes by $h \leftrightarrow n$, then[6] $u(h) = u(n)$, whatever the formal meaning of ”the agent believes $h \leftrightarrow n$” is. His proof is arguably not correct[7]. However, Schick can be credited with having proved the following interesting proposition:

Proposition 1. *Let Φ be the set of formulas of some propositional language*[8]*. Let $u : \Phi \to \mathbb{R}$ represent preferences on Φ. Define conditional utility by $u(h,k) = u(h \wedge k)$. Let $h \equiv h'$ denote logical equivalence*[9] *of $h, h' \in \Phi$. Then, if, for all $h, h' \in \Phi$,*

$$h \equiv h' \Rightarrow u(h) = u(h'),$$

then, for all $h, n \in \mathscr{F}$,

$$u(h, h \leftrightarrow n) = u(n, n \leftrightarrow h).$$

Proof. By equation (1),

$$u(h, h \leftrightarrow n) = u(h \wedge (h \leftrightarrow n)).$$

But $h \wedge (h \leftrightarrow n) \equiv h \wedge n$, therefore

$$u(h \wedge (h \leftrightarrow n)) = u(h \wedge n),$$

hence

$$u(h, h \leftrightarrow n) = u(h \wedge n).$$

Interchanging h and n,

$$u(n, n \leftrightarrow h) = u(n \wedge h).$$

Now, $u(h \wedge n) = u(n \wedge h)$, therefore

$$u(h, h \leftrightarrow n) = u(n, n \leftrightarrow h).$$

[6]Schick thus seems to identify the notion of co-reportiveness or co-referentiality and the notion of logical equivalence entailed by the bi-conditional. However these notions are not identical. For instance, the sentence “the triangle has three angles” and “the square has four angles” are logically equivalent, as they are tautologies; however, they are not co-reportive: they do not have the same factual interpretations in any reasonable relevant world. On the other hand, two co-reportive sentences are logically equivalent: if h and h' have the same factual interpretation in all relevant worlds, as they are true if and only if the fact, or state of affairs, they refer to obtains, they are always true at the same time and false at the same time. For instance, the sentences “The president of France in 2005 wears glasses” and “Jacques Chirac wears glasses” are co-reportive and they are logically equivalent: they must be true at the same time.

[7]Details on this could be provided to the interested reader.

[8]We mean by that a classical propositional language.

[9]By the logical equivalence of two sentences h and h', we here mean that for every truth assignment, h is true if and only if h' is true. Equivalently, $h \equiv h'$ if and only if “$h \leftrightarrow h'$” is a tautology.

$\square$

The formulation of extensionality by the formula

$$u(h, h \leftrightarrow n) = u(n, n \leftrightarrow h)$$

seems, in any case, according to the interpretation of conditional utility, a more rigorous way of formally stating the idea that if an agent believes that h and n are ”co-reportive”, then he of she attaches the same utility to both propositions.

The result presented in the previous proposition can, in fact, be interpreted as giving sufficient conditions for the Fregean notion of extensionality extended to preferences to be satisfied. Let us outline this interpretation. Interpret unconditional utility as experienced utility, or ex post utility, i.e. what the decision maker will effectively get if the proposition happens to be true (Kahneman and Tversky, 1984; Kahneman and Snell, 1990; Frisch, 1993; Kahneman, Wakker, and Sarin, 1997), and the conditional utility as the decision utility he or she uses to make his or her decision (same references), and that, therefore, incorporates his or her beliefs. Then, what the proposition says is that, under the presupposition that attributing a certain degree of decision utility to h given knowledge of k is the same as experiencing the conjunction of h and k, whenever the decision maker always experiences the same utility for two logically equivalent propositions, then he or she will attribute the same decision utility to propositions he or she believes to be co-reportive, therefore satisfying extensionality. Now, Frege's notion of intensionality implies that there is intensionality, i.e. the decision maker does not substitute *salva veritate*, or, for that matter, *salva utilitate*, two logically equivalent propositions, only if he or she does not see that these two sentences are co-reportive, i.e. only if their logical equivalence is not part of his or her beliefs. This is the conclusion of the proposition proved here[10]. Therefore, this result provides sufficient conditions for the extension of Frege's notion of (logical) extensionality to a decision context.

[10]Strictly speaking, this is not true as the language chosen by Schick is not expressive enough to allow for taking the contrapositive of the conclusion. A more expressive formulation would go as follows: let $\mathscr{K}$ be the set of all non empty finite subsets of Φ. An element $K \in \mathscr{K}$ represents the background knowledge or beliefs of the decision maker. Let $u(h, K)$ denote the utility of proposition h conditional on knowledge K. Then, Frege's notion of extensionality could be defined as follows:

$$(h \leftrightarrow n) \in K \Rightarrow u(h, K) = u(n, K).$$

This formulation has the advantage of allowing to consider the contrapositive of this condition, which corresponds to the way we have characterized Frege's notion of extensionality. It is not difficult to show that conditions very similar to those identified by Schick guarantee this form of extensionality.

2.2.2 The Second Derivation

Schick's alleged second derivation of extensionality can also be seen as being in fact another derivation of the Fregean notion of extensionality. We shall directly state what Schick has actually proved.

Proposition 2. *Let Φ be the set of formulas of some propositional language. Let $u : \Phi \to \mathbb{R}$ represent preferences on Φ and p be a probability measure representing beliefs on Φ. Let $h \equiv h'$ denote logical equivalence of $h, h' \in \Phi$. Then, if, for all $h, h' \in \Phi$,*

$$h \equiv h' \Rightarrow u(h) = u(h'),$$

and for all $h, k \in \Phi$,

$$u(h) = u(h \wedge k)p(k \mid h) + u(h \wedge \neg k)p(\neg k \mid h),$$

where $p(k \mid h)$ is the conditional probability of k given h, then, for all $h, n \in \Phi$,

$$p(h \leftrightarrow n \mid h) = p(n \leftrightarrow h \mid n) = 1 \Rightarrow u(h) = u(n).$$

Proof. If $p(h \leftrightarrow n \mid h) = 1$, then $u(h) = u(h \wedge (h \leftrightarrow n)) = u(h \wedge n)$. Similarly, $u(n) = u(n \wedge h) = u(h \wedge n) = u(h)$, as required. $\square$

Arguably, this is another way of expressing the fact that extensionality takes place only if the decision maker is *aware* of the co-reportiveness of the two propositions. In the first proposition, awareness is expressed by the fact that this equivalence is included in the beliefs of the decision maker through the conditional utility. In the second proposition, the beliefs of the decision maker are explicitly formalized by a probability measure, and being aware of co-reportiveness is expressed by the fact that it is attributed probability 1. The proposition thus gives sufficient conditions that guarantee this form of extensionality, among which the reader may have recognized Bolker and Jeffrey's formulation of expected utility. In other words, this theorem says that the Fregean notion of extensionality is a consequence of Bolker and Jeffrey's axioms.

Because Schick seeks a way of escaping from extensionality, even in the weaker form of Fregean extensionality, he proposes to replace the principle of expected utility by a weaker form using conditional utility. However, it is easy to show that this does not preclude an awkward form of Fregean extensionality:

Proposition 3. *Let Φ be the set of formulas of some propositional language. Let $v : \Phi \to \mathbb{R}$ represent preferences on Φ and $u : \Phi \times \Phi \to \mathbb{R}$ represent conditional preferences. Let*

p be a probability measure representing beliefs on Φ such that $p(k) = 1$ if and only if $k \equiv \top$, where $h \equiv h'$ denotes logical equivalence of $h, h' \in \Phi$. Then, if, for all $h, h' \in \Phi$,

$$h \equiv h' \Rightarrow v(h) = v(h'),$$

and for all $h, k \in \Phi$,

$$v(h) = u(h,k)p(k \mid h) + u(h, \neg k)p(\neg k \mid h),$$

where $p(k \mid h)$ is the conditional probability of k given h, then, for all $h, n \in \Phi$,

$$p(h \leftrightarrow n) = 1 \Rightarrow u(h, h \leftrightarrow n) = u(n, h \leftrightarrow n).$$

Proof. If $p(h \leftrightarrow n) = 1$, then $p(h \leftrightarrow n \mid h) = 1$ and $v(h) = u(h, h \leftrightarrow n)$. For the same reason, $v(n) = u(n, h \leftrightarrow n)$. But $p(h \leftrightarrow n) = 1$ implies $h \leftrightarrow n \equiv \top$, i.e. $h \equiv n$, therefore $v(h) = v(n)$, and this delivers the conclusion. □

Let us first comment on the assumptions of the proposition. The assumption that $p(k) = 1$ if and only if k is a tautology is true in the Bolker-Jeffrey decision model, which is the natural formal environment of Schick's discussion. It says that the agent cannot believe something that isn't true, that erroneous beliefs are excluded. It is akin to the truth axiom in the S5 model of epistemic logic. The weakened expected utility form that relates conditional and unconditional utility is the one advocated by Schick (albeit not because it is more intuitive than the stronger form but only on the ground that it precludes extensionality, a property that is not warranted given the proposition we have just proved).

Let us turn to the interpretation of the result. This interpretation is more difficult than in the previous case but it may go as follows: when some form of truth axiom and of expected utility hold, then if an agent believes that h and n are co-reportive ($p(h \leftrightarrow n) = 1$) and if the agent is aware of this belief (as formalized by the fact that his or her utility is conditioned on this belief), then he or she will attribute the same utility to h and n.

We may remark that the expected utility property can be replaced by the following weaker condition: if $k \equiv k' \equiv \top$, then for all h, $u(h, k) = u(h, k') = v(h)$; it amounts to the definition of unconditional utility from conditional utility.

2.3 Why is Intensionality Precluded in the Bolker-Jeffrey Model

As we have seen in the previous subsection, when Schick has tried to derive extensionality from axioms of utility theory, he has only partially succeeded: he has only proved some forms of Fregean extensionality. The reason for this is that he has, more or less unconsciously, used what, in our opinion, is the true statement of extensionality, namely:

$$\forall h, h' \in \Phi, h \equiv h' \Rightarrow u(h) = u(h'). \tag{2}$$

As co-reportive sentences are logically equivalent, this property will rule out framing effects. What are the axioms of utility theory that deliver it?

When reviewing the basic features of the Bolker-Jeffrey model, we alluded to the fact that the interpretation of elements of a Boolean algebra as propositions was not completely straightforward, and that it was related to extensionality. We shall now make this point clearer. Let Φ denote the set of formulas of a classical propositional language. The conjunction and disjunction operators, $\wedge$ and $\vee$, and the negation operator $\neg$, together with their classical properties, are natural candidates to make of Φ a Boolean algebra. However, in classical logic, all the properties of Boolean algebras are satisfied with the restriction that all equality signs in their statement must be replaced by the bi-conditional $\leftrightarrow$. Therefore, strictly speaking Φ is not a Boolean algebra. However it is very close to be one. Let us see how the connection is made. Let T, the set of classical truth functions, be the set of functions $t : \Phi \rightarrow \{0, 1\}$ satisfying $t(a \wedge b) = \min\{t(a), t(b)\}$ and $t(\neg\, a) = 1 - t(a)$. We say that a is logically equivalent to b, denoted $a \equiv b$ whenever $t(a) = t(b)$ for all $t \in T$. This defines an equivalence relation. Let $[a]$ be the equivalence class of a. $[a]$ is the *proposition expressed by* the sentence a. Let $\Pi := \Phi/\equiv$ be the set of propositions. Defining

$$[a] \wedge [b] := [a \wedge b],$$

$$[a] \vee [b] := [a \vee b],$$

and

$$\neg[a] := [\neg a]$$

makes Π into a Boolean algebra, the Boolean algebra of classical logic. Assuming the appropriate additional properties for Φ and T, one would easily ensure that this Boolean algebra be complete and atom free.

This construction shows that, if one identifies a formula and an element of a Boolean algebra, one is in fact implicitly neglecting any morphological differences between logically equivalent sentences or formulas, as it identifies a sentence with its equivalence class, the proposition that it expresses. This is what is done in the Bolker-Jeffrey decision model and this automatically yields extensionality in the strong sense defined above.

Another way of understanding this is to view extensionality as an implicit axiom of the theory: let us consider a preference relation $\succsim$ on Φ. In order to be able to apply Bolker's theorem to derive utilities and probabilities on *sentences*, one must first transfer the preference relation from Φ to Π to construct them on *propositions* . A natural way of doing this is defining:

$$[a] \succsim [b] \Leftrightarrow a \succsim b,$$

but this relation is ill-defined on Π unless one assumes the following consistency requirement:

$$a \equiv b \Rightarrow a \sim b.$$

This is exactly the extensionality principle.

3 Bypassing Extensionality in the Bolker-Jeffrey Decision Model

3.1 The Main Idea

As we have just seen, extensionality in the Bolker-Jeffrey decision model is forced onto us by an implicit axiom implied both by the mathematical structure of the set on which preference is defined, and by the interpretation of this set as a set of propositions. In order to bypass extensionality, therefore, it is desirable to modify either this structure or its interpretation in an appropriate way. By appropriate, we mean essentially that this modification should meet the following requirements.

1. Preserve the interpretation of objects of choice as propositions;
2. Preserve the mathematical tractability (i.e. applicability of Bolker's theorem);
3. Allow for a formal definition of extensionality in the model.

The first two requirements amount essentially to the preservation of the Boolean structure, and we shall see in the formal development how this is easily done. Some words of comment on the third requirement are however in order. If one wants to be able to bypass extensionality, one must first make this axiom explicit. However, if one defines extensionality of preferences in the same spirit as logical extensionality, imposing indifference for logically equivalent sentences, this is impossible in the Bolker-Jeffrey setup, as logically equivalent sentences are collapsed into a single object: the proposition they express. It is possible however to use another property of the logically equivalent sentences involved in framing effects: the fact that they refer to the same state of affairs[11]. Following

[11]This concept seems to fit also well the Fregean conception of extensionality: it is because "the morning star" and "the evening star" denote the same star that they should be substitutable *salva veritate*.

Schick, we called this property co-reportiveness. We can recast the usual definition of extensionality as applied to decision contexts as requiring indifference between co-reportive sentences. In fact, in the framework we propose, we will define the notion of co-reportive *propositions*, and define extensionality by the fact that they should be indifferent to the decision maker. This definition is in line with the classical definition of framing effects that we recalled at the beginning, according to which framing effects arise when different descriptions of the *same* problem, i.e. co-reportive sentences, yield divergent decisions, but we generalize it to allow for co-reportive propositions. As we shall see, this definition is, in turn, tractable in the Bolker-Jeffrey decision model.

In the definition of co-reportiveness, we have used the concept of state of affairs, that we should define a little bit more. The notion of a state of affairs requires the concept of an external world the state of which the language is designed to describe. A state of affairs is a certain relative disposition of the objects of the world, whatever these may be. States of affairs may be interpreted as ”events”, as in probability theory, i.e. things that may happen; said differently, a state of affairs is the set of all states of the world in which a given fact is true. This suggests that the set of states of affairs is a Boolean algebra. The set of propositions describing states of affairs is also a Boolean algebra, thanks to the identification of classes of logically equivalent sentences. Wittgenstein, in the *Tractatus*, has suggested that propositions are the isomorphic image of states of affairs. This further suggests that the fact that some proposition p describes a state of affair e can be formalized by the fact that p is the image of e by some isomorphism f. However, in order to allow for framing effects in our framework, we shall relax the assumption that f is an isomorphism an require only that it be a morphism. We indeed want to allow for the possibility of there being more conceivable propositions than states of affairs, a necessary condition for the existence of framing effects. Co-reportiveness of two sentences p and p' can thus be defined by the fact that there exists a state of affairs e and two morphisms f and f' such that $p = f(e)$ and $p' = f'(e)$.

These are the ideas that will guide us in trying to by-pass extensionality in the Bolker-Jeffrey decision model. As we shall see, the strategy adopted here will lead to a precise formalization of McKenzie's ideas as presented in the first section.

3.2 The Formal Model

We shall consider two sets $\mathscr{E}$ and $\mathscr{P}$, that we assume, like in the Bolker-Jeffrey decision model, to be atom free and complete Boolean algebras with maximal elements $\top$ and $\bot$. Elements of $\mathscr{E}$ are the possible states of affairs (events) of interest in a given decision problem. Elements of $\mathscr{P}$ are to be interpreted as propositions describing a given state of affairs, which in turn is an element of $\mathscr{E}$. To make this connection formal, we consider the set $\mathscr{F}$ of *frames*. A frame shall be formalized by a morphism[12] from $\mathscr{E}$ to $\mathscr{P}$. So $\mathscr{F}$ is a set of (not necessarily all) morphisms from $\mathscr{E}$ to $\mathscr{P}$. In order to make the problem meaningful, i.e. in order to establish a connection between propositions and events, we must make the following assumption:

Structural Assumption. *The set $\mathscr{F}$ is non-empty.*

A more demanding assumption would be the following:

Richness Assumption. *There exists an injective morphism in $\mathscr{F}$.*

This assumption would imply that $\mathscr{E}$ be isomorphic to a subset of $\mathscr{P}$, i.e. that the set $\mathscr{P}$ is sufficiently rich for any states of affairs to be describable in at least one frame by a proposition and for any two different states of affairs to be describable in this frame by two different propositions. It would therefore be appropriate to model the phenomenon of framing effects. However, this assumption is not used in the proof of the main result in this section, so that it is not necessary to impose it. We just mention it for the sake of completeness, and because it might appear useful in further developments of the theory we propose of framing effects.

For any $p \in \mathscr{P}$, if there exist a frame f and a state of affairs e such that $p = f(e)$, e will be said to be the *reference* of proposition p relative to frame f and f the *sense* of p relative to e.

We now assume that there is a binary relation $\succsim$ on the set $\mathscr{P}' = \mathscr{P} \setminus \{\perp\}$ of propositions without the null proposition.

Let us now give an example to illustrate how the formalism fits the intuition of framing effects, and more precisely how a frame and a proposition are conceptually different. Consider first the familiar example of the half-empty/hall-full glass. The state of affairs corresponding to this glass could be identified with the quantity of water the glass contains and the quantity it can contain. This is e. Now this state of affairs can be described

[12]By a morphism between two Boolean algebras $\mathscr{A}$ and $\mathscr{B}$, we mean a mapping

$$f : \mathscr{A} \to \mathscr{B}$$

such that:

$$f(a \wedge a') = f(a) \wedge f(a')$$

and

$$f(\neg a) = \neg f(a).$$

It follows from this definition that

$$f(a \vee a') = f(a) \vee f(a')$$

and that

$$f(\perp) = \perp \text{ and } f(\top) = \top.$$

according to two different frames: a positive frame, f^+, according to which it is half-full, and a negative frame, f^-, according to which it half-empty. Consider now the proposition "The glass is half-full" (s^+). This proposition can be interpreted as being $f^+(e)$, i.e. the state of affairs e described in the positive frame. Conversely, the proposition s^-: " the glass is half-empty" can be viewed as $f^-(e)$. We can now consider the same glass with half as much water, state of affairs e'. Now $f^+(e')$ is "the glass is one quarter-full" and $f^-(e')$ is "the glass is three quarters-empty".

Let us take another example that is more clearly related to a decision problem. This an example by Quattrone and Tversky (2000). Consider economic policies characterized by employment, unemployment and inflation rates. Policy A yields 90% of employment and 12% of inflation. Policy B yields 95% of employment and 17% of inflation. Policy C yields 10% of unemployment and 12% of inflation. Policy D yields 5% of unemployment and 17% of inflation. The problem is to choose the best policy. Clearly, policies A and C can be seen as describing the same state of affairs. However, they describe it using different frames, the employment frame and the unemployment frame. The same can be said of B and D. Therefore, they can easily be expressed in the formalism proposed here.

In order to state the result that shows that extensionality has been by-passed and which constitutes a formal account of McKenzie's theory, we need to introduce the following concept.

Definition 1. *Let e be a state of affairs. A* choice-relevant piece of information for e *is a state of affairs k such that:*

(i) $f(e \wedge k) \sim f'(e \wedge k)$*, for all* $f, f' \in \mathscr{F}$*;*

(ii) $f(e \wedge \neg k) \sim f'(e \wedge \neg k)$*, for all* $f, f' \in \mathscr{F}$*;*

(iii) $f(e \wedge k) \succ f(e \wedge \neg k)$ *for all* $f \in \mathscr{F}$*.*

Let us comment a little bit on this definition. Intuitively, a choice-relevant piece of information for a given decision is an information we'd rather have prior to making the decision and that helps us make the right decision. Now, from the formal definition of the concept of choice-relevant information, it is clear that prior to deciding whether he should go for decision e (i.e., decide that e be the true state of affairs), the decision maker would like to know if k obtains or does not obtain, because, if he or she knew it, he or she would know if he or she would be better off by choosing e or by refraining from doing so: he or she is better off when choosing e if and only if k obtains, irrespective of the frame (property (iii)). Moreover, this is an information that really concerns e because in fact the utility that the decision-maker will experience in the case that he or she has chosen e and that k obtains (respectively does not obtain) shall be the same irrespective of the frame.

Therefore, knowing if k is true is really knowing something about real consequences of the decision. In some sense k is an argument in favor of e: if k is true, one should go for e; if k is not true, one should refrain from doing so.

Let us give an example to illustrate this notion. Consider again policies C and D in the previous example. In the spirit of McKenzie's experiments, where he has shown that the frame chosen usually emphasizes the dimension that has increased relative to the usual situation, it could be inferred from the fact that the policy is described in terms of employment that the corresponding level of inflation is usually associated to a lower level of employment. On the contrary, framing this policy in terms of unemployment suggests this level of inflation usually leads to a lower rate of unemployment. Now, knowing which one of these statements is true is choice-relevant. Formally, let k be the state of affairs "a 12% rate of inflation usually corresponds to a rate of employment less than 90% and a rate of unemployment greater than 10%". Let e be the state of affairs described by both A and C. Clearly, $e \wedge k$ is preferred to $e \wedge \neg k$. Therefore, first k can be considered to be choice relevant.

Let $\mathscr{K}(e)$ be the set of choice-relevant pieces of information for e. The following theorem holds:

Theorem 2. *If $\succsim$ satisfies axioms BJ1 to BJ4, and the structural assumption, for each $f \in \mathscr{F}$, there exists a (finitely additive) probability measure P_f on $\mathscr{E}$ such that, for all $f, f' \in \mathscr{F}$, for all $e \in \mathscr{E}$, for all $k \in \mathscr{K}(e)$,*

$$f(e) \succsim f'(e) \Leftrightarrow P_f(k \mid e) \geq P_{f'}(k \mid e).$$

Proof. The proof proceeds in several steps.

Step 1. By Bolker's theorem, there exists a function $U : \mathscr{P} \to \mathbb{R}$ and a probability measure $P : \mathscr{P} \to [0, 1]$ such that

1. $U(p) \geq U(p') \Leftrightarrow p \succsim p'$.

2. For all $p, p' \in \mathscr{P}$, $U(p) = U(p \wedge p')P(p' \mid p) + U(p \wedge \neg p')P(\neg p' \mid p)$,

where

$$P(p' \mid p)\frac{P(p' \wedge p)}{P(p)}.$$

Step 2. Let $f \in \mathscr{F}, e \in \mathscr{E}$. Define:

$$P_f(e) := P(f(e)).$$

Let us show that P_f is a finitely additive probability measure.

First, because f is a morphism, $f(\top) = \top$, therefore, $P_f(\top) = P(\top) = 1$.

Second, for disjoint e and e',

$$P_f(e \vee e') = P(f(e \vee e')) = P(f(e) \vee f(e')) = P(f(e)) + P(f(e')) = P_f(e) + P_f(e').$$

Step 3. For all $e \in \mathscr{E}$, $k \in \mathscr{K}(e)$, $f, f' \in \mathscr{F}$, we have, using the fact that f and f' are morphisms:

$$\begin{aligned}
& f(e) \succsim f'(e) \\
& \Leftrightarrow U(f(e)) \geq U(f'(e)) \\
& \Leftrightarrow U(f(e) \wedge f(k))P(f(k) \mid f(e)) + U(f(e) \wedge \neg f(k))P(\neg f(k) \mid f(e)) \geq \\
& \qquad U(f'(e) \wedge f'(k))P(f'(k) \mid f'(e)) + U(f'(e) \wedge \neg f'(k))P(\neg f'(k) \mid f'(e)) \\
& \Leftrightarrow U(f(e \wedge k))P_f(k \mid e) + U(f(e \wedge \neg k))P_f(\neg k \mid e) \geq U(f'(e \wedge k))P_{f'}(k \mid e) + U(f'(e \wedge \neg k))P_{f'}(\neg k \mid e) \\
& \Leftrightarrow P_f(k \mid e) \geq \frac{U(f'(e \wedge k)) - U(f'(e \wedge \neg k))}{U(f(e \wedge k)) - U(f(e \wedge \neg k))} P_{f'}(k \mid e) + \frac{U(f'(e \wedge \neg k)) - U(f(e \wedge \neg k))}{U(f(e \wedge k)) - U(f(e \wedge \neg k))} \\
& \Leftrightarrow P_f(k \mid e) \geq P_{f'}(k \mid e),
\end{aligned}$$

where the last two lines follow from the fact that k belongs to $\mathscr{K}(e)$. $\square$

As it is easy to see, this theorem offers a way of by-passing extensionality. It gives indeed the conditions under which two co-reportive propositions are given the same utility by the decision maker. These conditions, which need not hold, are exactly the ones suggested by McKenzie in his defense of the normative status of framing effects. McKenzie's idea is that different frames provide different clues to the relevant information. Therefore, if, two logically equivalent, in McKenzie's words, i.e two co-reportive propositions are not substitutable as far as preferences are concerned, in other words, if they are not indifferent, it is because the frames in which their common reference is described do not convey the same probabilistic information about the relevant pieces of information. This is exactly what the theorem says: the two propositions $f(e)$ and $f'(e)$ are co-reportive, as they have the same reference, e. Now, the theorem says that two co-reportive propositions are indifferent if and only if the frames in which their common reference is expressed convey exactly the same information about choice-relevant pieces of information.

Let us come back to the preceding example to illustrate. As we said, it can be inferred from the speaker's choice of A that k is more likely than not $\neg k$. Similarly, it can be inferred from the speaker's choice of C that $\neg k$ is more likely than k. Let e be their common state of affairs and f^e and f^u be the employment and unemployment frames. Then $A = f^e(e)$ and $C = f^u(e)$. We therefore have that $P_{f^e}(k \mid e) > \frac{1}{2}$ and $P_{f^u}(k \mid e) < \frac{1}{2}$, therefore $P_{f^e}(k \mid e) > P_{f^u}(k \mid e)$, yielding $A \succ C$. It must be noticed, however, that direct comparisons of the same state of affairs framed in different ways have not been tested, to our knowledge, in the literature, although, as it can be seen, predictions could be made on this comparison based on McKenzie's theory. This is especially interesting if one wants to compare it with Prospect Theory (PT). PT makes predictions for behavior within a given frame, the gain frame or the loss frame. It does not make any prediction across frames. In that sense, they are therefore both partial theories of framing effects and they cannot be directly tested one against the other. So in principle we would need a combination of both to account for framing under risk. However, McKenzie's theory is a theory for all kinds of framing effects, not only for framing under risk. It is therefore a more promising theory, and the more comprehensive theory should probably be based on it.

4 Concluding Remarks

Framing effects, i.e. the fact that co-reportive sentences are attributed different utility values, are usually understood as demonstrating that subjects violate one of the most basic axioms of decision theory, namely extensionality. What this axiom really is, how it is to be understood, in what sense framing effects violate it and to what extent this violation is to be viewed as irrational are questions that have not, in our opinion, been thoroughly addressed in the literature. In this paper, we have attempted to contribute to this discussion in the framework of the Bolker-Jeffrey decision model.

We first show, basing our discussions on the contributions of two authors, Schick and McKenzie, why framing effects, i.e. intensionality in decision contexts, may be normatively grounded. The main reason is that contrary to what is usually asserted, it can be justified to set different utility values on co-reportive sentences, to the extent that these sentences provide different clues as to the likelihood of choice-relevant pieces of information. We discuss the philosophical implications of this intuition, and show how framing effects are the result of the consistency of pragmatic inferences and actual choice. Choice stems from the prior cognitive process of information construal in a perfectly rational way.

We then show that, in the standard Bolker-Jeffrey decision model, this cognitive adequacy of pragmatical inferences and choice is precluded, because extensionality follows basically from an implicit structural assumption embedded in the interpretation as propositions of the objects on which preferences are expressed in this model.

In order to bypass extensionality in this model, we introduce some slight modification of the structure of the underlying set and of the interpretation thereof, allowing for a precise definition of co-reportiveness. This modification leads to a formalization of McKenzie's intuition discussed in the first section, therefore showing that this intuition follows directly from a rigorous analysis of the modified Bolker-Jeffrey decision model that we propose.

This research leads to several important questions.

First, the philosophical discussion sketched here paves the way to a more general one on the nature and the status of cognitive illusions and anomalies in decision theory. It has been taken for granted in the last two decades that, contrary to, say, violations of expected utility like the Ellsberg paradox, these anomalies could not be incorporated into a rational account of decision making, and were evidence to the fact that a normatively sound and descriptively accurate decision theory was impossible. It seems to us that the results presented here cast some doubt on this claim, and give some hope that these anomalies may be treated as rigorously and successfully as the Ellsberg paradox and ambiguity aversion have been in the last decades. At a deeper level, though, the question of the proper norm that should be used to classify anomalies seems to us an open and interesting question.

Second, we would like to deepen our understanding of the way one could elude extensionality in the Bolker-Jeffrey decision model, doing away with this implicit axiom altogether. It seems to us that this would entail working with structures more general than Boolean algebras, which from a technical point of view is far from trivial. Moreover, in our representation theorem, there is no primitive definition of informational equivalence: informational equivalence is a revealed notion. It would be interesting to explore, from a syntactical point of view, this notion in propositional logic.

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