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Iterated Belief Revision and Conditional Logic

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Abstract. In this paper we propose a conditional logic called *IBC* to represent iterated belief revision systems. We propose a set of postulates for iterated revision which are a small variant of Darwiche and Pearl's ones. The conditional logic *IBC* has a standard semantics in terms of selection function models and provides a natural representation of epistemic states. We establish a correspondence between iterated belief revision systems and *IBC*-models. Our representation theorem does not entail Gärdenfors' Triviality Result.

1. Introduction

Belief revision and conditional logics have been studied in Artificial Intelligence, Logic and Philosophy [1, 3, 11, 12, 15, 16, 18]. A possible connection between belief change and conditionals was first suggested by Ramsey [16] who proposed that in order to decide whether to accept the conditional sentence "if A then B ", one should add the antecedent A to his belief set, changing it as little as possible, and then consider whether the consequent B follows.

In [5] Gärdenfors has suggested to formalize Ramsey's proposal within belief revision theory by the well known Ramsey Test:

$$A > B \in K \text{ iff } B \in K * A,$$

where K represents a belief set (that is, a deductively closed set of sentences) and $*$ represents a *belief revision operator*. The operator $*$, ruled by the AGM postulates, transforms ("revises") a belief set K by adding a formula A in such a way that the resulting belief set, denoted by $K * A$, is consistent if so is A ; moreover, $K * A$ is obtained by minimally changing K .

The semantics of belief revision and the semantics of conditionals are rather similar, since they can be both formulated in terms of possible worlds structures with a selection function. However, the Ramsey Test entails a principle, the *Monotonicity Principle* (if $K \subseteq K'$, then $K * A \subseteq K' * A$) that is incompatible with the *Preservation Principle* (if A is consistent with K , then $K \subseteq K * A$) which rules belief revision. The incompatibility between these two principles leads to the



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well known Triviality Result [5], that claims that no significant belief revision system is compatible with the Ramsey Test.

In [6] we have devised a correspondence between belief revision systems and conditional logic that, differently from the Ramsey Test, does not entail the Triviality Result. We have established a mapping between revision systems and the conditional logic BC by a representation result showing how each belief revision system determines a BC -structure, and how each BC -structure defines a belief revision system.

In this paper we show that a similar correspondence with conditional logics can be obtained for *iterated* belief revision systems. Iterated belief revision has been widely investigated in recent years [2, 3, 11]. In particular, it has been shown that the AGM postulates are too weak to ensure the rational preservation of revision strategies during the revision process. For this reason, new postulates have been proposed which “characterize belief revision as a process which may depend on elements of an epistemic state that are not necessarily captured by a belief set” [3]. These elements can be thought as revision strategies. The postulates should describe not only how the belief set changes throughout revision, but also how the revision strategies change. The question then is: when we learn new information do we have to keep the revision strategies we had before, or do we have to change them according to the new information? Boutilier [2] has proposed a belief revision operator, called *natural revision*, which guarantees that, after a revision step, the revision strategies are preserved as much as the AGM postulates permit. In [3] Darwiche and Pearl show that this solution may lead to counterintuitive results, since it might compromise the preservation of propositional beliefs. They adopt a more cautious approach which aims to preserve all those revision strategies that might not compromise the preservation of propositional beliefs. The postulates we consider are a slight strengthening of Darwiche and Pearl’s ones.

We then introduce a conditional logic IBC to represent iterated belief revision. The logic IBC is an extension of the logic BC and it provides a natural representation of epistemic states that are not introduced as new semantic objects, as it is done, for instance, by Friedman and Halpern in [4]. IBC -models are defined as standard possible worlds models with a selection function. Each possible world carries with itself, so to say, all the information concerning an epistemic state: a belief set and a set of revision strategies. As pointed out in [3], “any such strategy encodes, and is equivalent to, a set of “conditional” beliefs, that is, beliefs that one is prepared to adopt conditioned on any hypothetical evidence”. We identify an epistemic state with a set

of *equivalent* worlds in an *IBC*-structure, and the revision strategies relative to that state simply with the conditional formulas holding in those worlds.

We show, by means of a representation theorem, that each belief revision system determines an *IBC*-structure, and, under one additional condition, that each *IBC*-structure defines an iterated belief revision system. Differently from the Ramsey Test, our representation theorem does not run into the Triviality Result. The reason is that our representation theorem does not entail the Monotonicity Principle. We avoid monotonicity as we represent each epistemic state by a different equivalence class of worlds. In this way, there cannot be two epistemic states Ψ, Ψ' such that $\Psi \neq \Psi'$ and $\Psi \subseteq \Psi'$ (if one equivalence class is included in another, they are the same).

In the next section we introduce the set of postulates for iterated belief revision we will refer to, and we provide a concrete operator satisfying our postulates. In section 3 we present the conditional logic *IBC* and its semantics. In section 4 we prove a representation theorem which establishes a mapping between iterated revision and conditional models. In section 5, we discuss the representation result in light of the triviality problem.

2. Iterated Belief Revision

Alchourrón, Gärdenfors and Makinson have proposed a set of rationality postulates that any belief change operator must satisfy. In [1] they introduce the operations of *expansion* and *revision* on *belief sets* (that is, deductively closed sets of propositional formulas). *Expansion* is the simple addition of a formula A to a belief set K , and it is defined by: $K + A = Cn(K \cup \{A\})$. *Revision* is the consistent addition of a formula A to a belief set K , denoted by $K * A$.

As pointed out in the introduction, several authors [3, 4] have realized that AGM postulates are too weak to account for iterated revision: they only rule the preservation of propositional formulas while they do not say anything about the preservation of revision strategies.

In order to deal with iterated revision, we need a notion of *epistemic state*. An epistemic state has an associated belief set and an associated set of revision strategies that the agent wishes to employ in that state to accommodate new evidences. The revision strategies can be regarded as conditional beliefs, and they can be different in two different epistemic states even when they share the same belief set. In this paper we will only consider the revision of *consistent* epistemic

states. This restriction is embodied in the definition of iterated revision system.

DEFINITION 2.1. An *iterated belief revision system* is a triple $\langle \mathbf{S}, *, [\] \rangle$ where \mathbf{S} is a non-empty set whose elements are called epistemic states, $*$ is a *partial* function of type $*$: $\mathbf{S} \times \mathcal{L} \rightarrow \mathbf{S}$, and $[\]$ is a function of type $[\]$: $\mathbf{S} \rightarrow P(\mathcal{L})$ that maps each epistemic state to a belief set. We assume that for every formula A , $\Phi * A$ is *defined* if and only if Φ is consistent, i.e. the belief set $[\Phi]$ is consistent. The function $*$ is called a *revision operator* and it satisfies the postulates (R*1)–(R*6) and (I1)–(I4) listed below. Throughout the paper, when we write an expression such as $\Phi * A_1 * \dots * A_i$, we implicitly assume that it is defined, and therefore that each A_j , for $j = 1, \dots, i - 1$, is consistent. For $A, B \in \mathcal{L}$, $\Phi, \Psi \in \mathbf{S}$, the operator $*$ satisfies the following postulates:

$$(R^*1) \ A \in [\Psi * A]$$

$$(R^*2) \ \text{If } \neg A \notin [\Psi] \text{ then } [\Psi * A] = [\Psi] + A;$$

$$(R^*3) \ \text{If } A \text{ is satisfiable then } [\Psi * A] \text{ is also satisfiable};$$

$$(R^*4) \ \text{If } A_1 \equiv A_2 \text{ then } \Psi * A_1 = \Psi * A_2;$$

$$(R^*5) \ [\Psi * (A \wedge B)] \subseteq [\Psi * A] + B$$

$$(R^*6) \ \text{If } \neg B \notin [\Psi * A] \text{ then } [\Psi * A] + B \subseteq [\Psi * (A \wedge B)]$$

$$(I1) \ [\Psi * A * B] \subseteq [\Psi * B] + A;$$

$$(I2) \ \text{If } B \models \neg A \text{ then } [\Psi * A * B] = [\Psi * B];$$

$$(I3) \ \text{If } A \in [\Psi * B] \text{ then } [\Psi * A * B] = [\Psi * B];$$

$$(I4) \ \Psi * \top = \Psi$$

In spite of the different notation, postulates (R*1)–(R*6) correspond rather immediately to the AGM postulates (K1)–(K8), with the possible exception of (R*4) that deals with epistemic states rather than with belief sets. Postulate (R*4) claims that the revision of an epistemic state by two equivalent formulas A_1 and A_2 leads to the same epistemic state, whereas the AGM postulate (K4) only asserts that the epistemic states resulting from the revision by two equivalent formulas have the same belief set.

Postulates (I1)–(I3) are reformulations of Darwiche and Pearl's postulates for iterated revision [3]. They impose some constraints on the result of successive revisions of an epistemic state with two possibly related pieces of information. Postulate (I1) is a generalization of AGM

postulate (K3) (namely, $[\Psi * A] \subseteq [\Psi] + A$, which is comprised in (R*2)). The postulate claims that the belief set obtained by revising an epistemic state first with one formula and then with another is a subset of the belief set obtained by revising it with one of the two formulas and then adding the other.

Postulate (I2) says that if two contradictory pieces of information, A and B , are successively learnt, then the belief set obtained by the revision of Ψ by A and then by B does not depend on the intermediate revision A . In particular, (I2) implies that $[\Psi * A * \neg A] = [\Psi * \neg A]$.

Postulate (I3) says that if the formula A is less informative than B with respect to Ψ (in the sense that it can be derived from $\Psi * B$), then the belief set obtained by the revision first with A and then with B is the same as the belief set obtained by the revision of Ψ directly with B .

Finally, according to postulate (I4), the revision of an epistemic state with a tautology \top does not affect the epistemic state.

The following proposition shows that, given (R*1)-(R*6), postulates (I1)–(I3) are equivalent to Darwiche and Pearl's postulates (DP1)–(DP4) listed below:

- (DP1) If $B \models A$ then $[\Psi * A * B] = [\Psi * B]$;
- (DP2) If $B \models \neg A$ then $[\Psi * A * B] = [\Psi * B]$;
- (DP3) If $A \in [\Psi * B]$ then $A \in [\Psi * A * B]$;
- (DP4) If $\neg A \notin [\Psi * B]$ then $\neg A \notin [\Psi * A * B]$.

PROPOSITION 2.2. *Given postulates (R*1)–(R*6), we have that (DP1), (DP2), (DP3) and (DP4) are equivalent to (I1), (I2) and (I3).*

In spite of the previous result, we cannot prove that postulates (R*1)–(R*6), (I1)–(I4) are equivalent to the *whole set* of Darwiche and Pearl's postulates. Indeed, (I4) and (R*4) cannot be derived from Darwiche and Pearl's postulates since none of Darwiche and Pearl's postulates enforces the equality between epistemic states obtained through different revisions.

To conclude this section, we show that there is a non-trivial iterated belief revision system satisfying our postulates. More precisely, we show that Spohn's revision operator satisfies our postulates for iterated revision and that it defines a non-trivial iterated belief revision system.

We say that an iterated belief revision system $\langle \mathbf{S}, *, [] \rangle$ is *non-trivial* if there is at least one epistemic state $\Phi \in \mathbf{S}$ and three propositions A, B, C which are pairwise (logically) disjoint, such that $[\Phi] + A$, $[\Phi] + B$, $[\Phi] + C$ are consistent.

THEOREM 2.3. *There is at least one non-trivial iterated belief revision system.*

Proof. Fix a propositional language \mathcal{L} . Given the set W of all propositional evaluations for \mathcal{L} , consider the structure $\langle \mathbf{K}, *, [\] \rangle$, where $\mathbf{K} = \{k : W \rightarrow Ord\}$ is a set of functions from W to the class of ordinals such that for any k , $\exists w : k(w) = 0$. These functions are called *rankings* and can be seen as a natural representation of epistemic states. Let us define $k(A) = \min\{k(w) \mid w \models A\}$ for $A \in \mathcal{L}$, and $Prop(S) = \{A \in \mathcal{L} \mid \forall w \in S, w \models A\}$ for $S \subseteq W$. The operator $*$ is defined as follows¹: $k * A(w) = k(w) - k(A)$ if $w \models A$, $k * A(w) = k(w) + 1$ otherwise. $[\]$, of type $\mathbf{K} \rightarrow P(\mathcal{L})$, is defined by $[k] = Prop(\{w : k(w) = 0\})$. As proved in [3], the operator $*$ satisfies Darwiche and Pearl's postulates. From proposition 2.2 we can conclude that all our postulates, with the exception of postulates (I4) and (R*4), can be derived from Darwiche and Pearl's ones. It is sufficient to show that the operator $*$ satisfies (I4) and (R*4) to conclude that it satisfies all our postulates. As far as (I4) is concerned, $k = k * \top$ since for any w , $w \models \top$, and $k(\top) = 0$.

Concerning (R*4), if $A_1 \equiv A_2$ then, for any w , we have $w \models A_1$ iff $w \models A_2$. Therefore, $k(A_1) = k(A_2)$ and if $w \models A_1$, then $k * A_1(w) = k(w) - k(A_1) = k(w) - k(A_2) = k * A_2(w)$. On the contrary, if $w \not\models A_1$ then $k * A_1(w) = k(w) + 1 = k * A_2(w)$. We can therefore conclude that $\langle \mathbf{K}, *, [\] \rangle$ is an iterated belief revision system.

Now, consider the language \mathcal{L} containing only the propositional variables p_1, p_2, p_3, p_4 . Let $A = \neg p_2 \wedge \neg p_3 \wedge p_4$; $B = \neg p_3 \wedge \neg p_4 \wedge p_2$; $C = \neg p_2 \wedge \neg p_4 \wedge p_3$.

Clearly, $\vdash_{PC} \neg(A \wedge B)$, $\vdash_{PC} \neg(A \wedge C)$ and $\vdash_{PC} \neg(B \wedge C)$.

We show that there exists an iterated belief revision system and an epistemic state k in it such that $[k] + A$, $[k] + B$ and $[k] + C$ are consistent. Given the set W of all the possible interpretations of \mathcal{L} , consider the iterated belief revision system $\langle \mathbf{K}, *, [\] \rangle$ in which \mathbf{K} is the set of all possible rankings on W . In \mathbf{K} there is the ranking k : $k(w) = 0$ iff $w \models p_1$. Since $[k] = \{p_1\}$, it follows that $[k] + A$, $[k] + B$ and $[k] + C$ are consistent. \square

3. The Conditional Logic *IBC*

In this section we introduce the conditional logic *IBC* that we will use in the following to provide a logical formalization of iterated belief revision systems.

¹ This is the simplified version of Spohn's function proposed in [3]

DEFINITION 3.1. (IBC-logic) The language $\mathcal{L}_>$ of logic *IBC* is an extension of the language \mathcal{L} of classical propositional logic obtained by adding the conditional operator $>$.

Let us define the following modalities:

$$\Box A \equiv \neg A > \perp \quad \Diamond A \equiv \neg(A > \perp).$$

We define the language of modal formulas \mathcal{L}_\Box as the smallest subset of $\mathcal{L}_>$ including \mathcal{L} and closed under $\neg, \wedge, \Box, \Diamond$. We assume that the conditional $>$ has higher precedence than the material implication \rightarrow .

The logic *IBC* contains the following axioms and inference rules:

(G I) (*CLASS*) All classical axioms and inference rules;

(*ID*) $A > A$;

(*RCEA*) if $\vdash A \leftrightarrow B$ then $\vdash (A > C) \leftrightarrow (B > C)$;

(*RCK*) if $\vdash A \rightarrow B$ then $\vdash (C > A) \rightarrow (C > B)$;

(G II) (*DT*) $(A \wedge C) > B \rightarrow (A > (C \rightarrow B))$, for $A, B, C \in \mathcal{L}$;

(*CV*) $\neg(A > \neg C) \wedge (A > B) \rightarrow ((A \wedge C) > B)$, for $A, B, C \in \mathcal{L}$;

(G III) (*BEL*) $(A > B) \rightarrow \top > (A > B)$;

(*REFL*) $(\top > A) \rightarrow A$;

(*EUC*) $\neg(A > B) \rightarrow A > \neg(\top > B)$;

(*TRANS*) $(A > B) \rightarrow A > (\top > B)$;

(G IV) (*MOD*) $\Box A \rightarrow B > A$, where $A \in \mathcal{L}_\Box$;

(*U4*) $\Box A \rightarrow \Box \Box A$, where $A \in \mathcal{L}_\Box$;

(*U5*) $\Diamond A \rightarrow \Box \Diamond A$, where $A \in \mathcal{L}_\Box$.

(G V) (*C1*) $(A > B > C) \rightarrow (B > (A \rightarrow C))$, $A \in \mathcal{L}_\Box, C \in \mathcal{L}$.

(*C2*) $\Box \neg(A \wedge B) \wedge \Diamond A \rightarrow [(A > B > C) \leftrightarrow (B > C)]$, $A \in \mathcal{L}_\Box, C \in \mathcal{L}$;

(*C3*) $B > A \rightarrow [(A > B > C) \leftrightarrow (B > C)]$, $A \in \mathcal{L}_\Box, C \in \mathcal{L}$;

In the above axioms, formulas of the form $\top > A$ have a special meaning: we interpret $\top > A$ as “*A is believed*”.

We have gathered the axioms in different groups. Axioms of **(G I)** are those of the basic conditional logic CK+ID. Axioms of **(G II)** define standard properties of the conditional operator [12, 15, 18]. Axioms of **(G III)** are motivated by the interpretation of $\top > A$ as “*A is believed*”. The other axioms of this group (the last two for $A = \top$) give to this belief operator the properties of an S5 modality. Similarly, axioms of **(G IV)** define a necessity operator \Box and give

it S5-properties. Axiom (MOD) governs the relation between \Box and the conditional operator. Axioms of **(G V)** encode our postulates for iterated revision (I1), (I2),(I3) by conditional axioms.

It is worth noticing that axioms (ID), (DT), (CV), (MOD) belong to Stalnaker's logic **C2** (see [15]). However, Stalnaker's logic contains also other axioms such as (MP), (CS) and (CEM); these axioms could be derived from the axiomatization above if we added the axiom $A \rightarrow (\top > A)$ ("everything true is believed"), that we clearly do not want.

Moreover, it must be noticed that we have put restrictions on some axioms, by requiring that they only hold for formulas ranging over \mathcal{L} rather than over $\mathcal{L}_>$. These restrictions are motivated by the fact that our logic is intended to model the revision postulates, and some of them only put requirements on the belief sets, but not on the epistemic states. This is also true for the axioms (CV) and (DT)², which are used to represent postulates (R*6) and (R*2).

We develop a semantics for the logic *IBC* in the style of standard Kripke-like semantics for conditional logics. Our structures are possible world structures equipped with a selection function [15]. The selection function, call it f , given a formula A and a world w , picks up the *most preferred* or closest worlds to w , denoted by $f(A, w)$, which satisfy A (if any). To evaluate a conditional $A > B$ in a world w we check if B holds in all worlds in $f(A, w)$. Different logics are obtained by imposing different conditions on the selection function.

DEFINITION 3.2. An *IBC-structure* M has the form $\langle W, f, [[\]]\rangle$, where W is a non-empty set, whose elements are called possible worlds, f is a function of type $\mathcal{L}_> \times W \rightarrow 2^W$ and is called a selection function, $[[\]]: \mathcal{L}_> \rightarrow P(W)$ is a valuation function satisfying the following conditions: $[[\perp]] = \emptyset$; $[[A \wedge B]] = [[A]] \cap [[B]]$; $[[\neg A]] = W - [[A]]$; $[[A > B]] = \{w : f(A, w) \subseteq [[B]]\}$. The above definition is extended to the classical connectives $\vee, \rightarrow, \leftrightarrow$, by the usual classical equivalences. For $S \subseteq W$, let $Prop(S) = \{A \in \mathcal{L}: S \subseteq [[A]]\}$. We assume that the selection function f satisfies the following properties:

(G I) (S-ID) $f(A, w) \subseteq [[A]]$;

(S-RCEA) if $[[A]] = [[B]]$ then $f(A, w) = f(B, w)$;

(G II) (S-DT) $Prop(f(A \wedge C, w)) \subseteq Prop(f(A, w) \cap [[C]])$, $A, C \in \mathcal{L}$;

(S-CV) if $f(A, w) \cap [[C]] \neq \emptyset$ then $Prop(f(A, w)) \subseteq Prop(f(A \wedge C, w))$, $A, C \in \mathcal{L}$;

² Because of the limitations on conditional axioms such as (CV) and (DT), we cannot define a sphere semantics for the logic *IBC*.

- (**G III**) (S-REFL) $w \in f(\top, w)$;
 (S-TRANS) if $x \in f(A, w)$ and $y \in f(\top, x)$ then $y \in f(A, w)$;
 (S-EUC) if $x, y \in f(A, w)$ then $x \in f(\top, y)$;
 (S-BEL) if $w \in f(\top, y)$ then $f(A, w) = f(A, y)$;
- (**G IV**) (S-MOD) If $f(B, w) \cap [[A]] \neq \emptyset$ then $f(A, w) \neq \emptyset$, $A \in \mathcal{L}_\square$;
 (S-UNIV) if $[[A]] \neq \emptyset$, there is a formula B such that $f(B, w) \cap [[A]] \neq \emptyset$, $A \in \mathcal{L}_\square$;
- (**G V**) (S-C1) if $y \in f(A, x)$ then $Prop(f(B, y)) \subseteq Prop(f(B, x) \cap [[A]])$, $A \in \mathcal{L}_\square$.
 (S-C2) if $[[A]] \cap [[B]] = \emptyset$ and $y \in f(A, x)$ then $Prop(f(B, x)) = Prop(f(B, y))$, $A \in \mathcal{L}_\square$;
 (S-C3) if $f(B, x) \subseteq [[A]]$ and $y \in f(A, x)$ then $Prop(f(B, x)) = Prop(f(B, y))$, $A \in \mathcal{L}_\square$;

We say that a formula A is true in an *IBC*-structure $M = \langle W, f, [[\]] \rangle$ if $[[A]] = W$, and that a formula is *IBC*-valid if it is true in every *IBC*-structure. For readability, we also use the notation $x \models A$ instead of $x \in [[A]]$. Given an *IBC*-structure $M = \langle W, f, [[\]] \rangle$, a set of formulas Γ and a formula A , we define $\Gamma \models_M A$ if for every $w \in W$, $w \models B$ for every $B \in \Gamma$ implies $w \models A$ i.e. $\bigcap_{B \in \Gamma} [[B]] \subseteq [[A]]$. We then define the entailment relation $\Gamma \models A$ iff for every *IBC*-structure M , $\Gamma \models_M A$. As expected, $\emptyset \models_M A$ means that A is true in M and $\emptyset \models A$ that A is valid.

In an *IBC*-structure M , we can define by means of the selection function f the *equivalence relation* \approx_f on the set of worlds W as follows: for all $w, w' \in W$, $w \approx_f w'$ iff $w' \in f(\top, w)$. The properties of \approx_f being reflexive, transitive and symmetric come from the semantic conditions (*S-REFL*), (*S-TRANS*) and (*S-EUC*) of the selection function f (and, more precisely, from the last two conditions by taking $A = \top$).

As a consequence of (*S-BEL*), all worlds in one equivalence class $[w]_{\approx_f}$ evaluate conditional formulas in the same way. Moreover, by (*S-EUC*) and (*S-TRANS*), the set $f(A, w)$ is an equivalence class in itself. We will see in the next section that each model M determines an iterated belief revision system just by considering the equivalence classes as epistemic states (plus the empty set which represents the inconsistent state), and the revision operator $*$ as the canonical extension of f on the equivalence classes.

Notice that, since $(A > C) \vee \neg(A > C)$ is a tautology, from (*EUC*) and (*TRANS*) we can conclude $(A > (\top > C)) \vee (A > \neg(\top > C))$, that is, C is either believed or non-believed in the most preferred A -worlds. This is the conditional excluded middle, (*CEM*), restricted to belief

formulas. While the presence of (CEM) in Stalnaker's logic causes the selection function to select a single world (i.e. $f(A, w) = \{w'\}$ for all A and w), when (CEM) is restricted to belief formulas (as in our logic), it determines the uniqueness of the belief set associated to the epistemic state $f(A, w)$.

The other semantic conditions are needed to represent our postulates. From (S-UNIV), which corresponds to (U4) and (U5), and from (S-MOD) we get the property: if $[[A]] \neq \emptyset$, then $f(A, w) \neq \emptyset$. This property is needed to model the revision postulate (R*3). The restrictions we have put on (MOD), (U4), (U5) are needed since we cannot accept that the above property holds for all formulas $A \in L_{>}$. Having this property for arbitrary A would correspond to being able to reach an epistemic state from any other epistemic state by a revision step. This cannot be done by means of an arbitrary revision operator. In general, given two epistemic states Φ_1 and Φ_2 , there may not exist a formula A such that $\Phi_2 = \Phi_1 * A$.

The semantic conditions (S-C1), (S-C2), and (S-C3) are associated with the axioms (I1), (I2) and (I3) for iterated belief revision.

The axiomatization of *IBC* is sound and complete with respect to the semantics introduced above.

THEOREM 3.3 (Soundness). *If a formula A is a theorem of *IBC* then it is *IBC*-valid.*

The soundness result can be generalized to deduction from a non-empty set of formulas Γ . By $\Gamma \vdash A$, we mean that there is a derivation of A from Γ in the usual sense³.

COROLLARY 3.4. *If $\Gamma \vdash A$ then $\Gamma \models A$.*

We come now to the completeness. We prove it for the entailment relation $\Gamma \models A$. If $\Gamma = \emptyset$, we have the result for *IBC*-valid formulas.

THEOREM 3.5 (Completeness). *If $\Gamma \models A$ then $\Gamma \vdash A$.*

Proof. By contraposition, we show that if $\Gamma \not\models A$, i.e. $\Gamma \cup \{\neg A\}$ is consistent, then $\Gamma \not\vdash A$. Let us fix the language $\mathcal{L}_{>}$. We can prove:

Fact 0 Every consistent subset of \mathcal{L} can be extended to a maximal consistent set of \mathcal{L} -formulas. We assume that the usual properties of maximal consistent sets are known (e.g. if X is maximally consistent then $D \in X$ or $\neg D \in X$, etc.). Thus if $\Gamma \cup \{\neg A\} \not\models \perp$ then there is a maximal consistent set of formulas X_0 such that $\Gamma \cup \{\neg A\} \subseteq X_0$. We define $M = \langle W, f, [[\]]^M \rangle$, as follows

³ A derivation of a formula A from Γ is a sequence of formulas $A_1, \dots, A_n = A$, such that for every i , $A_i \in \Gamma$, or A_i is a theorem, or A_i is obtained by modus ponens from A_k and $A_j = A_k \rightarrow A_i$, with $j, k < i$.

$$\begin{aligned}
W &= \{X \mid X \text{ is maximally consistent}\}, \\
f(B, X) &= \{Y \in W \mid \{C \in \mathcal{L}_> \mid B > C \in X\} \subseteq Y\}, \\
[[p]]^M &= \{X \in W \mid p \in X\}
\end{aligned}$$

One can prove the following facts.

Fact 1 For every formula $B \in \mathcal{L}_>$ and $X \in W$, $B \in X$ iff $X \in [[B]]^M$.

Fact 2 The structure M satisfies all conditions of definition 3.2, except (possibly) the condition (S-UNIV). As an example, we prove condition (S-CV), (S-BEL), and (S-C1). The other are similar and left to the reader. In the proof, we will tacitly make use of Fact 1.

(S-CV) Let $f(D, X) \cap [[C]]^M \neq \emptyset$ and $B \in Prop(f(D, X))$, where $D, C, B \in \mathcal{L}$. By hypothesis it cannot be $D > \neg C \in X$, thus we have $\neg(D > \neg C) \in X$; by hypothesis we also have $D > B \in X$. By (CV) we conclude that $D \wedge C > B \in X$. Thus $B \in Prop(f(D \wedge C, X))$.

(S-BEL) let $X \in f(\top, Y)$, we show that $f(D, X) = f(D, Y)$. By definition of f , we have to show that for any $Z \in W$, $\{B : D > B \in X\} \subseteq Z$ iff $\{B : D > B \in Y\} \subseteq Z$.

We actually prove that in the hypothesis $X \in f(\top, Y)$ we have $D > B \in X$ iff $D > B \in Y$, from which the result follows. Let $D > B \in Y$, we have $\top > (D > B) \in Y$ by (BEL). Thus $D > B \in X$. Conversely, if $D > B \notin Y$, $\top > (D > B) \notin Y$ by (REFL). This implies that $\neg(\top > (D > B)) \in Y$, whence $\top > \neg(\top > (D > B)) \in Y$ by (EUC). Thus, $\neg(\top > (D > B)) \in X$ and also $\neg(D > B) \in X$ by (BEL), whence $D > B \notin X$.

(S-C1) Let $Y \in f(A, X)$.

We must prove that $Prop(f(B, Y)) \subseteq Prop(f(B, X) \cap [[A]]^M)$.

Let $C \in Prop(f(B, Y))$, we have:

$C \in Prop(f(B, Y))$ implies $B > C \in Y$, so that $\top > B > C \in Y$ by (BEL). We then have $\neg(\top > B > C) \notin Y$ and hence $A > \neg(\top > B > C) \notin X$, as by hypothesis $Y \in f(A, X)$. Thus $\neg(A > B > C) \notin X$ by (EUC); we conclude $A > B > C \in X$ by maximality of X .

Since $A > B > C \in X$, by axiom (C1) we get $B > (A \rightarrow C) \in X$. This shows that $X \in [[B > (A \rightarrow C)]]^M$, i.e. $f(B, X) \subseteq [[A \rightarrow C]]^M = [[C]]^M \cup (W - [[A]]^M)$. Thus $f(B, X) \cap [[A]]^M \subseteq [[C]]^M$, and this implies $C \in Prop(f(B, X) \cap [[A]]^M)$.

The structure M does not necessarily satisfy the condition (S-UNIV). However we can define a substructure M_0 of M which is still an *IBC*-structure and satisfies the universality condition. By Fact 0, there exists $X_0 \in W$ such that $\Gamma \cup \{\neg A\} \subseteq X_0$. We define a binary relation on W , for $X, Y \in W$, let

$RXY \equiv \forall D \in \mathcal{L}_\square (\square D \in X \rightarrow D \in Y)$ and then we let

$W_0 = \{Y \in W \mid RX_0Y\}$. We first show that

- (i) if $Z \in W_0$ and $Y \in f(B, Z)$, then $Y \in W_0$,
- (ii) for all $Y, Z \in W_0$, RYZ holds,
- (iii) $X_0 \in W_0$.

Property (i) means that W_0 is closed with respect to f . To show (i), let $\Box D \in X$, then $\Box\Box D \in X$ by (U4), then $\Box D \in Z$; by (MOD) we obtain $B > D \in Z$, whence $D \in Y$.

For (ii), let RX_0Y and RX_0Z , we show that RYZ holds. Suppose $D \notin Z$, then $\Box D \notin X_0$, then $\neg\Box D \in X_0$ then $\Diamond\neg D \in X_0$, so that $\Box\Diamond\neg D \in X_0$ by (U5). Then $\Diamond\neg D \in Y$, and this implies that $\Box D \notin Y$. Finally one, we can show (iii) $X_0 \in W_0$: if $\Box D \in X_0$ then $\top > D \in X_0$ by (MOD) and $D \in X_0$ by (REFL).

We can then define a substructure $M_0 = \langle W_0, f_0, [[\]^{M_0} \rangle$ of M , where $f_0(B, Z) = f(B, Z)$ and $[[p]]^{M_0} = [[p]]^M \cap W_0$, in particular the definition of f_0 is correct by virtue of (i).

Fact 3 For each formula C , $[[C]]^{M_0} = [[C]]^M \cap W_0$. This is proved by induction on the form of C , the details are left to the reader.

Fact 4 M_0 satisfies all conditions of definition 2, in particular it satisfies the condition (UNIV). In order to check (UNIV), let D be a modal formula and suppose that for all formulas B and $Z \in W_0$ $f(B, Z) \cap [[D]]^{M_0} = \emptyset$, in particular we have $f(D, Z) \cap [[D]]^{M_0} = \emptyset$; this implies $f(D, Z) = \emptyset$, that is $D > \perp \in Z$, whence $\Box\neg D \in Z$. By (ii) we have that for every $Y \in W_0$, RZY holds, thus $\neg D \in Y$, i.e. $[[D]]^{M_0} = \emptyset$.

The proofs of the other properties are similar and left to the reader.

We can now conclude the completeness proof. If $\Gamma \cup \{\neg A\} \not\vdash \perp$ then $\Gamma \cup \{\neg A\} \subseteq X_0$. Since $X_0 \in W_0$, we have $X_0 \in \bigcap_{B \in \Gamma} [[B]]^{M_0}$ and $X_0 \notin [[A]]^{M_0}$. Thus $\Gamma \models A$ does not hold. \square

4. Conditionals and Iterated Revision

In this section, we show a correspondence between iterated belief revision systems and *IBC*-structures. Given an iterated belief revision system, we can represent it as an *IBC*-structure such that, each (consistent) epistemic state Ψ is represented by a set of worlds W_Ψ which is an equivalence class with respect to \approx_f . The set of worlds W_Ψ (and, actually, each world in this set) provides all information concerning the epistemic state Ψ , including both the belief set and the revision strategies associated with Ψ . More precisely, we have that the belief set $[\Psi]$ is the set of formulas $\{C \in \mathcal{L} : W_\Psi \subseteq [[C]]\}$ or, equivalently, the set of formulas $\{C \in \mathcal{L} : \forall w \in W_\Psi, w \models \top > C\}$ (since for all $w \in W_\Psi, w \models \top > C$ iff $W_\Psi \subseteq [[C]]$).

Furthermore, the revision strategies associated with the epistemic state

Ψ are represented by all the conditional formulas $A_1 > A_2 > \dots > B$ true at the worlds in W_Ψ (which are the same for all the worlds in W_Ψ).

Before providing a representation theorem which establishes a precise correspondence between belief revision systems and *IBC*-structures, let us give an intuitive idea of the correspondence.

Let us consider any world w_Ψ in W_Ψ . We can check if B belongs to the belief set associated with the revised epistemic state $\Psi * A$, by evaluating the conditional $A > B$ at w_Ψ . In fact, we can represent the epistemic state $\Psi * A$ by the set of worlds $f(A, w_\Psi)$, that is to say the set of the most preferred A -worlds with respect to the world w_Ψ . Moreover, we can represent the belief set $[\Psi] + A$, obtained by an expansion of Ψ with A , as the subset of $f(\top, w_\Psi)$ satisfying A , namely $f(\top, w_\Psi) \cap [[A]]$.

On the other hand, we will see that (under one additional condition which we will introduce later) each *IBC*-structure M can be regarded as a belief revision system, where each world w in the structure (or, better, its equivalence class) represents a consistent epistemic state Ψ_w in such a way that we can define

$$\begin{aligned} [\Psi_w * A] &= \{B : w \models A > B\} \text{ and} \\ [\Psi_w] + A &= \{B : w \models \top > (A \rightarrow B)\}. \end{aligned}$$

As we will see, this can be generalized to the iterated case, by relying on the fact that (by the semantic properties of the logic) the set of worlds in $f(A, w)$ is an equivalence class in itself.

The representation theorem given below will show that the revision operator $*$ defined from a given *IBC*-structure satisfies the AGM postulates. However, we need to assume one condition on *IBC*-structures to enforce postulate (R*3). The condition is the following: we say that an *IBC*-structure satisfies the *covering condition* if, for any consistent formula A , $[[A]] \neq \emptyset$.

Given an *IBC*-structure $M = \langle W, f, [[]] \rangle$, let $W/\approx_f = \{[w]_{\approx_f} : w \in W\}$ be the quotient of W with respect to the equivalence relation \approx_f .

THEOREM 4.1 (Representation Theorem).

- (1) *Given a belief revision system $\langle \mathbf{S}, *, [] \rangle$, there is an *IBC*-structure $M_* = \langle W, f, [[]] \rangle$ such that:
for every Ψ in \mathbf{S} , there exists $w \in W$ such that, for $A_1 \dots A_n, B \in \mathcal{L} : B \in [\Psi * A_1 * \dots * A_n]$ iff $w \models A_1 > \dots > A_n > B$.*
- (2) *Given an *IBC*-structure $M = \langle W, f, [[]] \rangle$ which satisfies the covering condition, there is an iterated belief revision system $\langle \mathbf{D}_M, *_M, [] \rangle$ such that:*

$$\begin{aligned} \mathbf{D}_M &= W/\approx_f \cup \{\emptyset\}; \\ [w]_{\approx_f} *_M A &= f(A, w); \\ [[w]_{\approx_f}] &= Prop([w]_{\approx_f}) \end{aligned}$$

and, for each $[w]_{\approx f} \in W/\approx f$, and $A_1 \dots A_n, B \in \mathcal{L}$

$$B \in [[w]_{\approx f} *_M A_1 \dots *_M A_n] \text{ iff } w \models A_1 > \dots A_n > B.$$

Proof. PART (1). Given a belief revision system $\langle \mathbf{S}, *, [] \rangle$, we define an *IBC*-structure $M_* = \langle W, f, [] \rangle$ as follows:

$$W = \{(\Psi, w) : w \text{ is a classical interpretation, } \Psi \in \mathbf{S} \text{ and } w \models [\Psi]\};$$

$$[[p]] = \{(\Psi, w) \in W : w \models p\} \text{ for all propositional letters } p \in \mathcal{L}.$$

$f(A, (\Psi, w))$ and $[[A]]$ can be defined by double induction on the structure of the formula A . At each induction step, for each connective \circ , $[[A \circ B]]$ is defined by making use of the valuation of the subformulas ($[[A]]$ and $[[B]]$) and of the selection function for subformulas (for instance, $f(A, w)$); moreover, $f(A \circ B, w)$ is defined by possibly making use of the valuation of the formula $A \circ B$ itself. In particular we let:

$$f(A, (\Psi, w)) = \{(\Psi', w') \in W : \Psi' = \Psi * A\}, \text{ if } A \in \mathcal{L};$$

$$f(A, (\Psi, w)) = \{(\Psi', w') \in W : \Psi' = \Psi * \phi_A\}, \text{ if } A \notin \mathcal{L}$$

and there exists a formula $\phi_A \in \mathcal{L}$ such that $[[A]] = [[\phi_A]]$;

$$f(A, (\Psi, w)) = \emptyset, \text{ otherwise.}$$

By making use of the properties of the revision operator $*$, we can show that M_* is an *IBC*-structure, that is, it satisfies all the semantic properties listed in definition 3.2. We show the most significant cases.

(S-ID) $f(A, (\Psi, w)) \subseteq [[A]]$.

By definition of f , for all $(\Psi', w') \in f(A, (\Psi, w))$ we have $\Psi' = \Psi * A$ and $w' \models [\Psi * A]$. Let $A \in \mathcal{L}$, then, by postulate (R*1) we have that $A \in [\Psi * A]$, and hence $(\Psi', w') \in [[A]]$.

The case of $A \notin \mathcal{L}$, but $[[A]] = [[\phi_A]]$ with $\phi_A \in \mathcal{L}$, is similar. In case $A \notin \mathcal{L}$ and there is no $\phi_A \in \mathcal{L}$ such that $[[A]] = [[\phi_A]]$, then $f(A, (\Psi, w)) = \emptyset$ and the property trivially holds.

(S-RCEA) if $[[A]] = [[B]]$ then $f(A, (\Psi, w)) = f(B, (\Psi, w))$.

Let $A, B \in \mathcal{L}$. If $[[A]] = [[B]]$ holds in M_* , then it must be that $A \equiv B$ is valid, as the set $\{w' : (\Psi', w') \in [[A]]\}$ contains all and only the classical models of A . By postulate (R*4) we have that $\Phi * A = \Phi * B$. Hence, if $(\Psi', w') \in f(A, (\Psi, w))$, then $\Psi' = \Psi * A = \Psi * B$ and from $w' \models [\Psi * A]$ we can conclude $w' \models [\Psi * B]$. Therefore, $(\Psi', w') \in f(B, (\Psi, w))$.

In case $A \notin \mathcal{L}$ but there is a formula $\phi_A \in \mathcal{L}$ such that $[[A]] = [[\phi_A]]$, the proof is the same.

If $A \notin \mathcal{L}$ and there is no formula $\phi_A \in \mathcal{L}$ such that $[[A]] = [[\phi_A]]$, then $f(A, (\Psi, w)) = \emptyset$. Moreover, as $[[A]] = [[B]]$, it must be that $B \notin \mathcal{L}$ and there is no formula $\phi_B \in \mathcal{L}$ such that $[[B]] = [[\phi_B]]$. Thus, also $f(B, (\Psi, w)) = \emptyset$.

(S-DT) $Prop(f(A \wedge C, w)) \subseteq Prop(f(A, (\Psi, w)) \cap [[C]])$, $A, C \in \mathcal{L}$.

Let $B \in Prop(f(A \wedge C, (\Psi, w)))$. For all $(\Psi', w') \in f(A \wedge C, (\Psi, w))$, $\Psi' = \Psi * (A \wedge C)$ and $w' \models [\Psi * (A \wedge C)]$. Since $f(A \wedge C, (\Psi, w))$ contains

all the classical models of $[\Psi * (A \wedge C)]$, we have that: $B \in Prop(f(A \wedge C, (\Psi, w)))$ if and only if for all w' , if $w' \models [\Psi * (A \wedge C)]$ then $w' \models B$. Hence, $B \in Prop(f(A \wedge C, (\Psi, w)))$ if and only if $B \in [\Psi * (A \wedge C)]$.

$B \in Prop(f(A, (\Psi, w)) \cap [[C]])$ if and only if $B \in [\Psi * A] + C$.

From (R*5) we have that $[\Psi * (A \wedge C)] \subseteq [\Psi * A] + C$, from which, given the above equivalences, the property trivially follows.

The proof of (S-CV) is very similar to the one of (S-DT). The proofs of (S-REFL), (S-TRANS), (S-EUC) and (S-BEL) are easy and left to the reader.

(S-MOD) $f(B, (\Psi, w)) \cap [[A]] \neq \emptyset$ implies $f(A, (\Psi, w)) \neq \emptyset$, where $A \in \mathcal{L}_\square$.

It can be easily seen that, if $A \in \mathcal{L}_\square$ then either $A \in \mathcal{L}$ or $[[A]] = [[\phi_A]]$ for some $\phi_A \in \mathcal{L}$.

Assume first that $A \in \mathcal{L}$. Then, either $[[A]] = \emptyset$ (and $f(B, w) \cap [[A]] = \emptyset$, so that the property holds trivially) or $[[A]] \neq \emptyset$. In the latter case, A is consistent. Then, by (R*3), $\Psi * A$ is consistent and, by construction, $f(A, (\Psi, w)) \neq \emptyset$.

In case $[[A]] = [[\phi_A]]$ for $\phi_A \in \mathcal{L}$, the proof is similar.

(S-UNIV) if $[[A]] \neq \emptyset$, then there is a formula B such that $f(B, (\Psi, w)) \cap [[A]] \neq \emptyset$, where $A \in \mathcal{L}_\square$.

As in the previous case, if $A \in \mathcal{L}_\square$ then either $A \in \mathcal{L}$ or $[[A]] = [[\phi_A]]$ for some $\phi_A \in \mathcal{L}$. In the first case, from $[[A]] \neq \emptyset$, we have by construction that $f(A, (\Psi, w)) \neq \emptyset$. As (S-ID) holds, the conclusion follows by taking $B = A$. In the second case, the proof is similar, by taking $B = \phi_A$.

(S-C2) if A is modal, $[[A]] \cap [[B]] = \emptyset$ and $(\Psi', w') \in f(A, (\Psi, w))$, then $Prop(f(B, (\Psi, w))) = Prop(f(B, (\Psi', w')))$.

We can distinguish three cases: the case when both $A, B \in \mathcal{L}$; the case when $A \notin \mathcal{L}$ or $B \notin \mathcal{L}$, but there is a $\phi_A \in \mathcal{L}$ ($\phi_B \in \mathcal{L}$) such that $[[A]] = [[\phi_A]]$ (respectively, $[[B]] = [[\phi_B]]$); the case when either $A \notin \mathcal{L}$ and there is no $\phi_A \in \mathcal{L}$ such that $[[A]] = [[\phi_A]]$, or $B \notin \mathcal{L}$ and there is no $\phi_B \in \mathcal{L}$, such that $[[B]] = [[\phi_B]]$.

In the first case, from the fact that $[[A]] \cap [[B]] = \emptyset$, it follows that $A \wedge B$ is inconsistent (otherwise, by (R*3), for any $\Psi \in \mathbf{S}$, $[\Psi * (A \wedge B)]$ would be satisfiable, which would imply that $(\Psi * (A \wedge B), w) \in W$, $[[A \wedge B]] \neq \emptyset$ and $[[A]] \cap [[B]] \neq \emptyset$). As $A \wedge B$ is inconsistent, $B \models \neg A$. By postulate (I2), then, we conclude that $[\Psi * A * B] = [\Psi * B]$. By the definition of f , it follows immediately that the property holds.

The case when there exists $\phi_A \in \mathcal{L}$ (or there exists ϕ_B in \mathcal{L}) such that $[[A]] = [[\phi_A]]$ (respectively, $[[B]] = [[\phi_B]]$) can be treated in a similar way.

In the third case, the property follows easily from the definition of f . If ϕ_A does not exist, $f(A, (\Psi, w)) = \emptyset$, i.e. there is no $(\Psi', w') \in$

$f(A, (\Psi, w))$ and the property follows trivially. If ϕ_B does not exist then $f(B, (\Psi, w)) = \emptyset$ and $f(B, (\Psi', w')) = \emptyset$.

The proofs of the semantic properties (S-C1) and (S-C3) are similar to that of (S-C2).

To conclude the proof of part (1), it is not difficult to show that the model M_* satisfies the condition that, for each epistemic state Ψ in \mathbf{S} , and for each $A_1, \dots, A_n, B \in \mathcal{L}$,

$B \in [\Psi * A_1 * \dots * A_n]$ if and only if $(\Psi, w) \models A_1 > \dots > A_n > B$.

PART (2). Let us consider an *IBC*-structure $M = \langle W, f, [\] \rangle$ satisfying the *covering condition*. For each world w in W , we introduce an epistemic state $[w]_{\approx f}$, and define the belief set associated to the epistemic state by $[[w]_{\approx f}] = Prop([w]_{\approx f})$.

We define the revision operator $*_M$ by stipulating that for each $[w]_{\approx f}$ and each $A \in \mathcal{L}$, $[w]_{\approx f} *_M A = f(A, w)$.

Notice that for the semantic properties of *IBC* if $f(A, w)$ is non empty then it is an equivalence class, namely the equivalence class $[w']_{\approx f}$ of any $w' \in f(A, w)$. We allow revision by an inconsistent formula, and such revision gives the inconsistent epistemic state. The empty set of worlds represents the inconsistent epistemic state. Thus, the set of epistemic states is given by $W/\approx_f \cup \{\emptyset\}$. This makes sense since we have: if A is inconsistent then $[w]_{\approx f} *_M A = f(A, w) = \emptyset$, and moreover $Prop(\emptyset) = \mathcal{L}$.

Concerning iterated revision, it follows by construction that $[w]_{\approx f} *_M A *_M B = f(B, w')$ for any $w' \in f(A, w)$.

We show that the revision system $\langle \mathbf{D}_M, *_M, Prop \rangle$, where $\mathbf{D}_M = W/\approx_f \cup \{\emptyset\}$ satisfies the postulates (R*1) – (R*6), (I1) – (I4).

(R*1) $A \in [[w]_{\approx f} *_M A]$.

Since $[w]_{\approx f} *_M A = f(A, w)$, we have that, for all $B \in \mathcal{L}$, $B \in [[w]_{\approx f} *_M A]$ if and only if $B \in Prop(f(A, w))$. By (S-ID), we have $f(A, w) \subseteq [[A]]$, and hence $A \in Prop(f(A, w))$, from which we conclude $A \in [[w]_{\approx f} *_M A]$.

(R*2) If $\neg A \notin [[w]_{\approx f}]$ then $[[w]_{\approx f} *_M A] = [[w]_{\approx f}] + A$.

We first prove the inclusion " \subseteq ". Let $B \in [[w]_{\approx f} *_M A]$. Then, $B \in Prop(f(A, w))$. By (S-DT), $Prop(f(A, w)) \subseteq Prop(f(\top, w) \cap [[A]])$. Thus, $B \in Prop(f(\top, w) \cap [[A]])$. But, $Prop(f(\top, w) \cap [[A]]) = [[w]_{\approx f}] + A$, and hence $B \in [[w]_{\approx f}] + A$.

We prove the inclusion " \supseteq ". Assume that $\neg A \notin [[w]_{\approx f}]$. Then, $f(\top, w) \cap [[A]] \neq \emptyset$. If $B \in [[w]_{\approx f}] + A$ then $B \in Prop(f(\top, w) \cap [[A]])$. It is easy to see that from (S-CV) it follows that $Prop(f(\top, w) \cap [[A]]) \subseteq Prop(f(A, w))$. Thus, $B \in Prop(f(A, w))$ and $B \in [[w]_{\approx f} *_M A]$.

(R*3) If A is satisfiable then $[[w]_{\approx f} *_M A]$ is also satisfiable.

Assume that A is satisfiable. Then, by the covering condition, $[[A]] \neq \emptyset$.

As a consequence of (S-UNIV) and (S-MOD) we have that $f(A, w) \neq \emptyset$. But $[w]_{\approx f} *_M A = f(A, w)$, whence $[w]_{\approx f} *_M A \neq \emptyset$.

(R*4) If $A_1 \equiv A_2$ then $[w]_{\approx f} *_M A_1 = [w]_{\approx f} *_M A_2$

Assume that $A_1 \equiv A_2$. Then $[[A_1]] = [[A_2]]$ and, by (S-RCEA) $f(A_1, w) = f(A_2, w)$. But $[w]_{\approx f} *_M A_1 = f(A_1, w)$ and $[w]_{\approx f} *_M A_2 = f(A_2, w)$, so that the thesis follows.

(R*5) $[[w]_{\approx f} *_M (A \wedge B)] \subseteq [[w]_{\approx f} *_M A] + B$.

Let $C \in [[w]_{\approx f} *_M (A \wedge B)]$. Then, $C \in Prop(f(A \wedge B, w))$. By (S-DT), $Prop(f(A \wedge B, w)) \subseteq Prop(f(A, w) \cap [[B]])$. Hence, $C \in Prop(f(A, w) \cap [[B]])$. But, $Prop(f(A, w) \cap [[B]]) = [[w]_{\approx f} *_M A] + B$, and hence $C \in [[w]_{\approx f} *_M A] + B$.

(R*6) If $\neg B \notin [[w]_{\approx f} *_M A]$ then $[[w]_{\approx f} *_M A] + B \subseteq [[w]_{\approx f} *_M (A \wedge B)]$.

Assume that $\neg B \notin [[w]_{\approx f} *_M A]$. Then, $f(A, w) \cap [[B]] \neq \emptyset$. If $C \in [[w]_{\approx f} *_M A] + B$ then $C \in Prop(f(A, w) \cap [[B]])$. It is easy to see that from (S-CV) it follows that $Prop(f(A, w) \cap [[B]]) \subseteq Prop(f(A \wedge B, w))$. Hence, $C \in Prop(f(A \wedge B, w))$ and $C \in [[w]_{\approx f} *_M (A \wedge B)]$.

(I1) $[[w]_{\approx f} *_M A *_M B] \subseteq [[w]_{\approx f} *_M B] + A$.

By (S-C1), we have that for any $w' \in f(A, w)$, $Prop(f(B, w')) = Prop(f(B, w) \cap [[A]])$. Since $[w]_{\approx f} *_M A *_M B = f(B, w')$, we can conclude that: if $C \in [[w]_{\approx f} *_M A *_M B]$ then $C \in Prop(f(B, w'))$. Thus, $C \in Prop(f(B, w) \cap [[A]])$. But then $C \in [[w]_{\approx f} *_M B] + A$.

(I2) If $B \models \neg A$ then $[[w]_{\approx f} *_M A *_M B] = [[w]_{\approx f} *_M B]$.

Assume that $B \models \neg A$. Then, $[[A]] \cap [[B]] = \emptyset$. Let $C \in [[w]_{\approx f} *_M A *_M B]$. Notice that A must be a consistent formula, as $[[w]_{\approx f} *_M A *_M B]$ is defined. Thus $f(A, w) \neq \emptyset$. By construction we have: $[w]_{\approx f} *_M A *_M B = f(B, w')$ for any $w' \in f(A, w)$. Hence, $C \in Prop(f(B, w'))$. By (S-C2) we have that $Prop(f(B, w')) = Prop(f(B, w))$ and, therefore, $C \in Prop(f(B, w))$. Thus, $C \in [[w]_{\approx f} *_M B]$. Similarly, from $C \in [[w]_{\approx f} *_M B]$ we conclude that $C \in [[w]_{\approx f} *_M A *_M B]$.

(I3) If $A \in [[w]_{\approx f} *_M B]$ then $[[w]_{\approx f} *_M A *_M B] = [[w]_{\approx f} *_M B]$.

Suppose that $A \in [[w]_{\approx f} *_M B]$. It follows that $f(B, w) \subseteq [[A]]$ and, by (S-C3), we can conclude that for any $w' \in f(A, w)$, $Prop(f(B, w)) = Prop(f(B, w'))$. Since $[w]_{\approx f} *_M A *_M B = f(B, w')$ for any $w' \in f(A, w)$, and $[w]_{\approx f} *_M B = f(B, w)$, we have that $[[w]_{\approx f} *_M A *_M B] = Prop(f(B, w'))$ and $[[w]_{\approx f} *_M B] = Prop(f(B, w))$. By (S-C3) it follows that $[[w]_{\approx f} *_M A *_M B] = [[w]_{\approx f} *_M B]$.

(I4) $[w]_{\approx f} *_M \top = [w]_{\approx f}$

Since by (S-REFL) it holds that $w \in f(A, w)$, we have that $[w]_{\approx f} *_M \top = f(\top, w) = [w]_{\approx f}$.

To conclude the proof of part (2), we observe that the property $B \in [\Psi *_M A_1 *_M \dots *_M A_n]$ if and only if $(\Psi, w) \models A_1 > \dots > A_n > B$ follows directly from the definition of $*_M$. \square

The following corollaries of the representation theorem give a characterization in the logic IBC of what can be derived by the revision of any epistemic state, independently from the specific properties of the revision system. Corollary 4.3 characterizes what can be derived from the revision of any epistemic state with a given belief set K . As a particular case, corollary 4.4 gives a characterization of what can be derived by the revision of an “empty” epistemic state, that is an epistemic state whose only beliefs are the tautologies. Corollary 4.5, on the other hand, gives a characterization of what can be derived by the revision of *any* epistemic state, no matter what is its associated belief set. For any set K of formulas in \mathcal{L} , we define the set $Th(K)$ of formulas in $\mathcal{L}_>$ as follows:

DEFINITION 4.2.

$$Th(K) = \{\top > C : C \in K\} \cup \{\neg(\top > C) : C \notin K\} \cup \{\diamond A : \not\vdash_{PC} \neg A\}.$$

Notice that $Th(K)$ depends non-monotonically on K , in the sense that from $K \subseteq K'$ it does not follow $Th(K) \subseteq Th(K')$. Thus from $Th(K) \vdash A_1 > \dots > A_n > B$ we cannot conclude that $Th(K') \vdash A_1 > \dots > A_n > B$. This reflects the non-monotonicity of the belief revision operator with respect to belief sets: if $[\Psi] \subseteq [\Psi']$ it does not follow that $[\Psi * A_1 \dots * A_n] \subseteq [\Psi' * A_1 \dots * A_n]$.

COROLLARY 4.3. For $A_1, \dots, A_n, B \in \mathcal{L}$, the following are equivalent

1. $Th(K) \vdash A_1 > \dots > A_n > B$
2. for all $\langle \mathbf{S}, *, [] \rangle$, for all $\Psi \in \mathbf{S}$ with $[\Psi] = K, B \in [\Psi * A_1 \dots * A_n]$.

Proof. (\Rightarrow) We show the contrapositive, i.e. that if there is a revision system $\langle \mathbf{S}, *, [] \rangle$, and $\Psi \in \mathbf{S}$ such that $[\Psi] = K$ and $B \notin [\Psi * A_1 \dots * A_n]$ then $Th(K) \not\vdash_{IBC} A_1 > \dots > A_n > B$.

Let $\langle \mathbf{S}, *, [] \rangle$ and $\Psi \in \mathbf{S}$ such that $[\Psi] = K$; let $B \notin [\Psi * A_1 \dots * A_n]$. By the representation theorem, there exists an IBC -model M , and there is a world w such that $\forall E, D_i \in \mathcal{L}, E \in [\Psi * D_1 \dots * D_n]$ iff $w \models D_1 > \dots > D_n > E$; Thus, by the hypothesis, we have:

- (1) $w \not\models A_1 > \dots > A_n > B$.

Since $\Psi = \Psi * \top$, we have:

- (2) $w \models \top > C$ holds for any $C \in [\Psi]$, and $w \models \neg(\top > C)$ for any $C \notin [\Psi]$.

Furthermore, we can show that:

- (3) $w \models \diamond A$ for all A such that $\not\vdash_{PC} \neg A$.

To see this, let A be such that $\not\vdash_{PC} \neg A$, we know (by (R*1) and (R*3)) that there is an epistemic state $\Psi \in \mathbf{S}$ such that $A \in [\Psi]$; by construction of M (see the proof of the representation theorem) there is $w \in W$

such that $w \models [\Psi]$, whence $[[A]] \neq \emptyset$; but this implies that $f(A, w) \neq \emptyset$ (by the universality condition). Therefore, $w \models \Diamond A$. By (2) and (3) we have $w \models Th(K)$ and thus by (1) $Th(K) \not\models A_1 > \dots > A_n > B$. By the soundness of IBC , it follows that $Th(K) \not\models A_1 > \dots > A_n > B$.

(\Leftarrow) Again, we show the contrapositive. Suppose that $Th(K) \not\models A_1 > \dots > A_n > B$. By the completeness of IBC , $Th(K) \not\models A_1 > \dots > A_n > B$ and there is M and $w \in M$ such that $w \models Th(K)$ and $w \not\models A_1 > \dots > A_n > B$. Since $w \models \{\Diamond A \in \mathcal{L} : \not\models_{PC} \neg A\}$, the model M satisfies the covering condition. Then, by the second half of the representation theorem, there is an iterated belief revision system $\langle \mathbf{D}_M, *_M, [\] \rangle$ and an epistemic state $\Psi \in \mathbf{D}_M$ such that $[\Psi] = Prop([w]_{\approx_f}) = \{C \in \mathcal{L} : w \models (\top > C) = K\}$, and such that $B \notin [\Psi * A_1 \dots * A_n]$. \square

To introduce the next corollary, we denote by TAU the belief set which contains exactly the classical propositional tautologies. We call *empty* an epistemic state Ψ if $[\Psi] = \text{TAU}$. Notice that in any IBC -model $M = \langle W, *, [\] \rangle$, $\forall w \in W$, $w \models Th(\text{TAU})$ if and only if $w \models \{\neg(\top > B) : \not\models_{PC} B\}$, therefore $Th(\text{TAU}) = \{\neg(\top > B) : \not\models_{PC} B\}$.

COROLLARY 4.4. *For all $A_1, \dots, A_n, B \in \mathcal{L}$, the following are equivalent:*

1. $Th(\text{TAU}) \vdash A_1 > \dots > A_n > B$
2. for all $\langle \mathbf{S}, *, [\] \rangle$, for all *empty* $\Psi \in \mathbf{S}$ $B \in [\Psi * A_1 \dots * A_n]$.

The last corollary characterizes what can be derived by the revision of any epistemic state in any belief revision system.

COROLLARY 4.5. *For all $A_1, \dots, A_n, B \in \mathcal{L}$ the following are equivalent:*

1. $\{\Diamond A : \not\models_{PC} \neg A\} \vdash A_1 > \dots > A_n > B$
2. for all $\langle \mathbf{S}, *, [\] \rangle$, for all $\Psi \in \mathbf{S}$ $B \in [\Psi * A_1 \dots * A_n]$.

The set of assumptions $\{\Diamond A \in \mathcal{L} : \not\models_{PC} \neg A\}$ is needed to restrict our consideration to models which satisfy the covering condition.

5. Triviality and Non Triviality

The representation theorem provided in section 4 establishes a relation between conditionals and belief revision which is reminiscent of the Ramsey Test. However, as a difference with the Ramsey Test, the relation established by the representation theorem does not entail the

triviality result, for it holds also for non-trivial systems ⁴.

Let us briefly analyze here why our representation theorem does not run into the triviality result. We recall that the triviality result, by [5], claims that there is no non-trivial belief revision system that satisfies the Monotonicity Principle. We also recall that the Monotonicity Principle is a direct consequence of the Ramsey Test.

Differently from the Ramsey Test, our representation theorem does not entail any form of monotonicity of the revision operator.

Given our mapping between *IBC*-structures and belief revision systems, let $[w]_{\approx f}$ and $[w']_{\approx f}$ be two equivalence classes that represent two different epistemic states. We can express Monotonicity as follows: if $[[w]_{\approx f}] \subseteq [[w']_{\approx f}]$ then $[[w]_{\approx f} * A] \subseteq [[w']_{\approx f} * A]$.

Let $[[w]_{\approx f} * A] = Prop(f(A, w))$ and $[[w']_{\approx f} * A] = Prop(f(A, w'))$. If $w' \notin [w]_{\approx f}$, there is no reason why $f(A, w) = f(A, w')$ or $Prop(f(A, w)) = Prop(f(A, w'))$ should hold. Thus Monotonicity does not hold.

One may think that in order to prevent the monotonicity of revision (and the consequent triviality of the belief revision system) it could be sufficient not to include conditionals in epistemic states. The next proposition shows that this is not the case.

PROPOSITION 5.1. *Given an iterated belief revision system $\langle \mathbf{S}, *, [] \rangle$, suppose there is a mapping $\mu : \mathbf{S} \rightarrow 2^{\mathcal{L}^>}$ and there is a model M^5 such that:*

1. $\forall \Psi, \Psi' \in \mathbf{S}$ if $[\Psi] \subseteq [\Psi']$ then $\mu(\Psi) \subseteq \mu(\Psi')$,
2. $\forall \Psi \in \mathbf{S}, B \in [\Psi * A]$ iff $\mu(\Psi) \models_M A > B$;

*then for any $\Psi, \Psi' \in \mathbf{S}$, $[\Psi] \subseteq [\Psi']$ implies $[\Psi * A] \subseteq [\Psi' * A]$. Thus $\langle \mathbf{S}, *, [] \rangle$ is trivial.*

Proof. Let $\langle \mathbf{S}, *, [] \rangle$, M and μ satisfy the hypothesis of the proposition; consider two epistemic states $\Psi, \Psi' \in \mathbf{S}$ such that $[\Psi] \subseteq [\Psi']$. If $B \in [\Psi * A]$ by 2. we have $\mu(\Psi) \models_M A > B$. Since $[\Psi] \subseteq [\Psi']$, by 1. we have $\mu(\Psi) \subseteq \mu(\Psi')$, thus we also have $\mu(\Psi') \models_M A > B$. By 2. this implies that $B \in [\Psi' * A]$. We have shown that $[\Psi * A] \subseteq [\Psi' * A]$. \square

Let us call *monotonic* a mapping μ satisfying the condition 1. of the above proposition. The proposition says that a non-trivial belief revision system cannot be represented in any conditional model by any monotonic mapping, no matter what are the properties of the model. In a looser sense, it says that we cannot hope to represent belief revision

⁴ Theorem 2.3 shows that there is at least one non-trivial iterated belief revision system

⁵ Not necessarily an *IBC* -model.

systems in conditional logics by associating to epistemic states sets of conditional formulas which depend *monotonically* on the belief sets associated to the states.

On the other hand, we *can* represent any non-trivial belief revision system in some conditional model (namely, an *IBC*-model) if we give up the monotonicity condition on the mapping μ . This is an easy consequence of the first half of the representation theorem.

PROPOSITION 5.2. *For any iterated belief revision system $\langle \mathbf{S}, *, [] \rangle$ there is a mapping $\mu : \mathbf{S} \rightarrow 2^{\mathcal{L}^>}$ and there is an *IBC* model M such that: $\forall \Psi \in \mathbf{S}, B \in [\Psi * A]$ iff $\mu(\Psi) \models_M A > B$.*

Proof. Given $\langle \mathbf{S}, *, [] \rangle$ define for every $\Psi \in \mathbf{S}$:

$$\begin{aligned} \mu(\Psi) = & \{A_1 > \dots > A_n > B : A_1, \dots, A_n, B \in \mathcal{L} \text{ and} \\ & B \in [\Psi * A_1 * \dots * A_n]\} \\ \cup & \{\neg(A_1 > \dots > A_n > B) : A_1, \dots, A_n, B \in \mathcal{L} \text{ and} \\ & B \notin [\Psi * A_1 * \dots * A_n]\} \end{aligned}$$

Let M_* be the *IBC* model as defined in the proof of the first half of theorem 4.1. We show that for any $\Psi \in \mathbf{S}, B \in [\Psi * A]$ iff $\mu(\Psi) \models_M A > B$. The (\Rightarrow)-direction is immediate by the definition of μ . To show the (\Leftarrow)-direction, assume that $\mu(\Psi) \models_M A > B$, but $B \notin [\Psi * A]$; we have $\neg(A > B) \in \mu(\Psi)$, thus $\mu(\Psi)$ is inconsistent. But, by the representation theorem there is a world w in M , such that

$$B \in [\Psi * A_1 * \dots * A_n] \text{ iff } w \models A_1 > \dots > A_n > B.$$

This implies that $w \models \mu(\Psi)$, whence $\mu(\Psi)$ must be consistent and we have a contradiction. \square

It is needless to say that the mapping μ in the above proof is non-monotonic.

6. Conclusions

In this paper we have presented the conditional logic *IBC* to capture iterated belief revision. The logic *IBC* provides a natural representation of epistemic states and belief sets. We have proved a representation result which establishes a correspondence between iterated belief revision systems and the models of our logic. We have shown that our mapping between revision and conditionals does not incur Gärdenfors' Triviality Result.

The postulates for iterated revision we have introduced are a slightly stronger reformulation of Darwiche and Pearl's ones [3] (see section 2).

Our postulates have some similarities with Lehmann's ones [11]. Lehmann represents the sequence of revisions applied to the initial belief set, which is assumed consistent, by a sequence of consistent formulas σ and denotes by $[\sigma]$ the resulting belief set. In his framework the revision of $[\sigma]$ by the formula A depends not only on the belief set $[\sigma]$, but also on the entire sequence of revisions σ . The sequence of revisions plays the role of the epistemic states in our context. Our postulates (I1), (I3), can be derived from his postulates but (I2) cannot. On the other hand, Lehmann's postulates (I4) and (I6) cannot be derived from our postulates.

The way we represent beliefs has some similarities with the necessity operator \Box_K introduced by Nejdil and Banagl in [14], which, however, is parametric with respect to a knowledge base K . In [14] the satisfiability of a formula is defined with respect to the model associated with a given belief set K , whereas the revision function is external to models and applies to them. As a difference, we incorporate the revision function in the models of our logic (namely in the selection function), since the aim of our proposal is to depart as little as possible from standard conditional logics and their model theory.

In [4] Friedman and Halpern introduce the notion of *belief change system* (BCS), a generalization of Gärdenfors's belief revision system. A BCS contains "three components: the set of possible *epistemic states*, a *belief assignment* that maps each epistemic state to a set of beliefs, and a *transition function* which transforms epistemic states. The properties of the transition function are expressed in terms of conditional axioms. As a difference with standard conditional logics and with our approach, Friedman and Halpern's semantics is built upon epistemic states rather than worlds as primitive semantic objects.

In [10] Katsuno and Satoh present a unifying view of nonmonotonic reasoning, belief revision and conditional logic based on the notion of minimality. More precisely, they introduce ordered structures and families of ordered structures as a common ingredient. *Ordered structures* are triples (W, \leq, V) containing a set W of worlds, a preorder \leq and a valuation function V . They provide a semantic model to evaluate unnested conditional formulas. *Families of ordered structures* are defined as collections of ordered structures, and their axiomatization corresponds to well known conditional logics, as VW, VC, and SS. In particular, Katsuno and Satoh [10] show that, given a revision operator $*$, for each belief set K , there is an ordered structure O_K (satisfying the covering condition) such that all the formulas in K are true in all minimal worlds of O_K (written $O_K \models K$), and $K * A = \{B : O_K \models A > B\}$.

Since an ordered model O_K contains a *single* ordering relation \leq , it can only represent a *single* belief set K and its revisions. Moreover, since ordered structures do not handle nested conditionals, iterated revision cannot be captured in this formalization.

In [13] Lindström and Rabinowicz show several possible solutions to the Ramsey Test problem. Our solution has some similarities with what they call the “indexical interpretation of conditionals”, according to which the truth value of a conditional formula in a world depends not only on the world, but also on the epistemic state in which the conditional appears. We have seen that also in our logic the evaluation of a conditional in a world depends on the epistemic state associated to that world. The difference between our proposal and Lindström and Rabinowicz’s is that we do not need to introduce epistemic states as extra elements in our semantics. Moreover, Lindström and Rabinowicz [13] do not aim to define a conditional logic system as we have done in this work.

Several authors have studied an alternative notion of belief change called “belief update” [9, 10, 8], which does not enforce the preservation principle. In [8] Grahne has proposed a conditional logic which combines update and counterfactual conditionals. The logic *IBC* is related to the logic proposed by Grahne if we made the conditional $T > A$ equivalent to A by adding the axiom $A \rightarrow (T > A)$ to our axiomatization. In such a case, some of our axioms (namely, (*BEL*) and (*TRANS*)) become tautological, while other axioms coincide with the axioms of Grahne’s logic. In particular, all axioms of Grahne’s logic can be derived from ours (provided we lift the restriction that some formulas have to range over \mathcal{L} rather than over $\mathcal{L}_>$).

In [17] Ryan and Schobbens establish a link between updates, which are regarded as existential modalities, and counterfactuals, which are regarded as universal modalities. The Ramsey rule is an axiomatization of the inverse relationship between the two sets of modalities. Again, if the axiom $A \rightarrow T > A$ is added to our axiomatization, all the counterfactual rules which (according to the mapping in [17]) correspond to the update postulates can be derived from our axioms (provided we lift the restriction that some formulas have to range over \mathcal{L}).

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