Deductive Reasoning Under Uncertainty: A Water Tank Analogy

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This is an updated version which replaces the previous versions

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Abstract

This paper describes a cubic water tank equipped with a movable partition receiving various amounts of liquid used to represent joint probability distributions. This device is applied to the investigation of deductive inferences under uncertainty. The analogy is exploited to determine by qualitative reasoning the limits in probability of the conclusion of twenty basic deductive arguments (such as Modus Ponens, And-introduction, Contraposition, etc.) often used as benchmark problems by the various theoretical approaches to reasoning under uncertainty. The probability bounds imposed by the premises on the conclusion are derived on the basis of a few trivial principles such as "a part of the tank cannot contain more liquid than its capacity allows", or "if a part is empty, the other part contains all the liquid". This stems from the equivalence between the physical constraints imposed by the capacity of the tank and its subdivisions on the volumes of liquid, and the axioms and rules of probability. The device materializes de Finetti's *coherence* approach to probability. It also suggests a physical counterpart of Dutch book arguments to assess individuals' rationality in probability judgments in the sense that individuals whose degrees of belief in a conclusion are out of the bounds of coherence intervals would commit themselves to executing physically impossible tasks.

Keywords:

Uncertain reasoning; inference schemas; de Finetti's coherence; probability logic; qualitative reasoning.

1. Introduction

The main objective of the present paper is to describe in a diagrammatic form a physical implementation of de Finetti's (1937) coherence based theory of probability applied to deduction with uncertain premises.

For a long time communicators have made use of various graphical representations that help acquire, process and mentally represent cross-classified categorical data (see Friendly, 2002, for an historical account). However helpful contingency tables may be, they do not count as diagrammatic representations because they are just typographical displays showing rows and columns of numbers that do not possess the analogical or computational qualities of diagrams. A crucial step is reached when the table is divided into rectangles whose areas are proportional to the quantities present in the joint distributions. This type of representation was developed in detail by Hartigan & Kleiner (1981) under the name of "mosaic", mainly as a tool to expose deviation from independence. It had been used earlier without elaboration by Bertin (1967 p. 256) and Edwards (1972, p. 47). Edwards' diagram (Figure 1) illustrates the joint distribution of boys and girls in a school crossed with long-haired and short-haired children. The fundamental characteristic of this diagram is that the areas of the various rectangles are proportional to the products of the marginal probabilities.

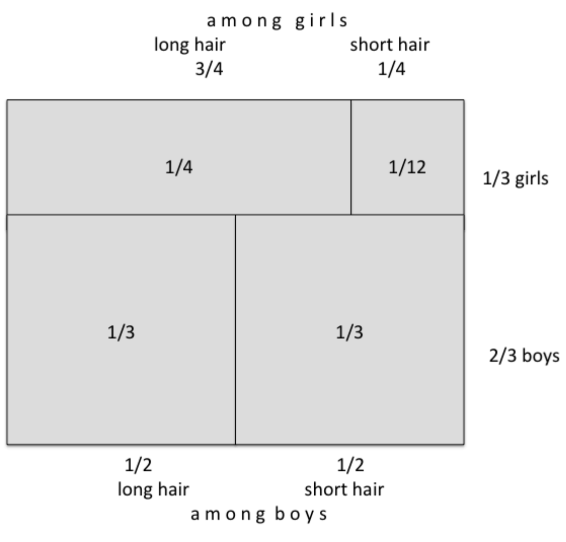


Figure 1. Edward's (1972) representation of a two-way distribution.

More recently, Oldford (2003a, 2003b; Oldford & Cherry, 2006) investigated the properties of diagrams of this type, which he called "eikosograms" (for "probability drawing"). He showed their usefulness to represent conditional probability, conditional and unconditional dependence or independence. Considering a wide range of puzzles (e.g., the prisoner's dilemma, the Monty Hall, Simpson's paradox), classic problems (e.g., the Engineer-and-Lawyer problem) and applications (e.g., medical tests), his demonstration of the effectiveness of eikosograms to represent the information and calculate the solution is deeply impressive. In these papers Oldford mentioned an interpretation of the diagrams in terms of a "water container metaphor" to infer a marginal value when the joint probabilities are known but did not pursue this idea any further. Moreover, the use of the diagrams was strictly limited to problem solving. In fact, the way in which mosaic representations are commonly used is static. They display sets of data in which all values (generally frequencies) are known or determined, and aim to help exhibit relations of dependence leading to the solution.

The present paper too uses a diagrammatic representation of probability that belongs to the mosaic type, but concerns itself with a different domain. It is not devoted to statistical or probabilistic problem solving, but to probabilistic logic, and it adopts a *dynamic* representation, in which the values in the diagram can vary. Using a two-compartment water tank presented in a diagrammatic form, it develops an analogy of the laws of probability and conditional probability, subsequently applied to deductive schemas of inference. It will be used to derive and illustrate an answer to the major question posed by deduction with uncertain premises: what degrees of belief is it rational to attribute to the conclusion of an inference, given the probability of the premises?

In the first section of the paper the basic diagram, the probabilistic interpretation of its components, and the representation of elementary laws of conditional probability are presented. In the main part it is shown how inference schemas are interpreted in terms of the analogy and how the probability of their conclusion is obtained by virtual operations on the device, while the validity of the inferences with respect to several criteria is systematically examined. The last section presents a brief discussion of the relationship between the physical principles used in the tank analogy and the axioms and rules of probability theory.

2. The water tank analogy

2.1.The basic diagram

A cubic tank has a movable partition (parallel to one side) that divides it into two compartments, left (A) and right (A'). In what follows, all the diagrams represent a vertical cross-section of the tank perpendicular to the partition (see Figure 2). Each compartment can contain some amount of liquid independently of the other.

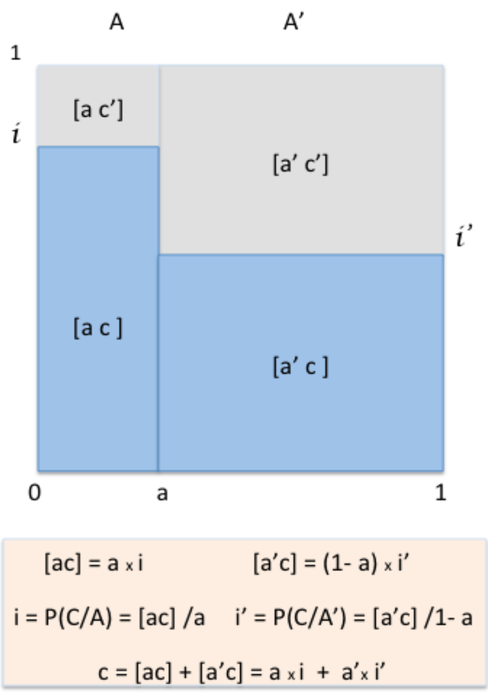


Figure 2. The basic analogy.

The dimensions of the tank are: height = 1 unit, width = 1 unit. The width of A is denoted by <u>a</u> ($0 \leq a \leq 1$). It is a measure of the capacity of A. The capacity of A' is measured by a' = 1-a.

The tank has a full capacity of 1 unit. Its overall content (occupying rate) is denoted by <u>c</u> ($0 \leq c \leq 1$).

The levels of liquid in A and A' are denoted by i and i', respectively. The volumes of liquid in A and A' are denoted by [ac] and [a'c], respectively. Similarly the volumes of the empty spaces in A and A' are denoted by [ac'] and [a'c'].

The contents (volumes of liquid) of A and A' are i x a and i'x a', respectively. i x a is a measure of [ac], and i'x a' of [a'c'].

2.2. Probabilistic interpretation

A is the event "the left compartment occupies the whole tank"; a is the probability of this event, that is, the degree to which the whole tank is occupied by A (which will also be denoted by P(A)).

Calling C the event "the tank is filled up with liquid", c is the probability of C, that is, the degree to which the tank is filled with liquid (or the volume of liquid in the tank), which will also be denoted by P(C)[1,2].

The event relative to compartment A, "A is filled up with liquid" is a *conditional event* denoted by C/A. This is a *conditional* event because the filling is *restricted* to the compartment A. i is the probability of this event, that is, the degree to which A is filled with liquid (which will be also denoted by P(C/A)). The level i of liquid in A represents the probability of de Finetti's conditional event C/A, that is, the probability of *if A then C.* (Similarly the level i' in A' represents the probability of the conditional event C/A', *if A' then C).*

The event relative to the liquid, "all the liquid is contained in A", is a conditional event denoted by A/C, its probability denoted by P(A/C) is the degree to which all of the liquid is in A, which is the probability of *if C then A.*

The joint probability of A and C, P(A AND C) is represented by the liquid common to A and C, that is, the volume of liquid in A, which is measured by a x i:

P(A AND C) = P(A) x P(C/A) (also denoted by [ac]).

Similarly, P(NOT-A AND C) = P(NOT-A) x P(C/ NOT-A) (also denoted by [a'c]);

P(A AND NOT-C) = P(A) x P(NOT-C/A) (also denoted by [ac']);

P(NOT-A AND NOT-C) = P(NOT-A) x P(NOT-C/ NOT-A) (also denoted by [a'c']).

The correspondence between the physical features constitutive of the water tank and the axioms of probability are the following:

1) The measures of volume or capacity are ≥ 0. This is the counterpart of the requirement that the probability of an event E defined on a sample space Ω be such that P(E) ≥ 0.

2) The tank (or a compartment) cannot contain more than its capacity. This is the counterpart of P(Ω) = 1 and P(A/A) = 1, respectively.

3) Given an amount of liquid in the tank (or in a compartment), adding some more liquid results in an amount whose volume is the sum of the two. This the counterpart of the axiom of additivity.

This extends the correspondence between the axioms of probability and the graphic features of eikosograms shown by Oldford & Cherry (2006).

2.3. Elementary rules

The water tank analogy allows an easy representation of the total probability rule. The water tank also allows a clear representation of Bayes' rule and probabilistic independence, as eikosograms do (Oldford & Cherry, 2006). With our current interpretation and notations, these are as follows.

2.3.1.The total probability rule

It is obtained by adding the contribution of each compartment to the whole (see Figure 2).

c = [ac] + [a'c] = (a x i) + (a' x i')

2.3.2. Bayes' rule

The amount of liquid in A, [ac], can be viewed as the part of A that is filled, a x P(C/A), or the part of liquid that is in A, c x P(A/C) (see Figure 3a), hence: a x P(C/A) = c x P(A/C), or:

P(A/C) = (a x i) / c

2.3.3. Representation of independence

The levels are the same in A and A': c = i = i', hence: c = [ac] / a = [a'c] / a' (see Figure 3b).

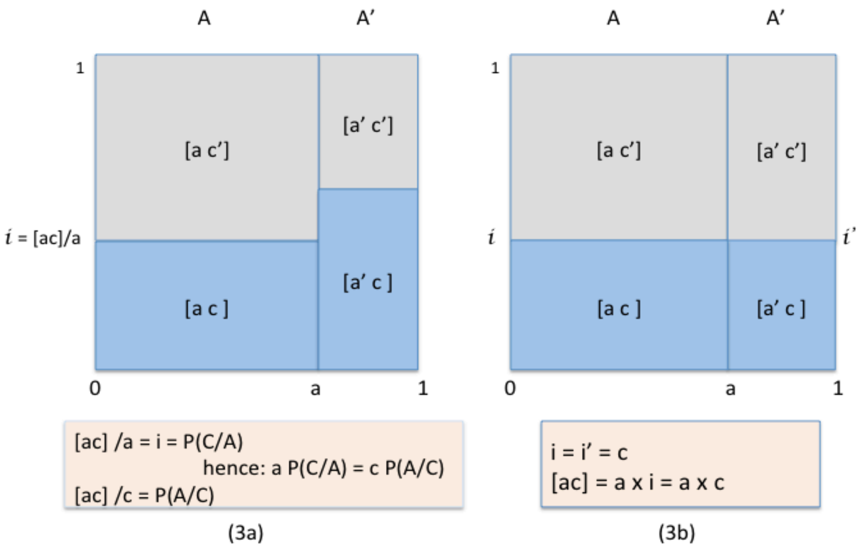


Figure 3. (a) Bayes' rule. (b) Independence

3. Deduction under uncertainty

The investigation of reasoning under uncertainty, more specifically of deductive reasoning with uncertain premises, has led to the elaboration of probabilistic logics by Adams (1998) and by Hailperin (1996) and independently to psychological studies (George, 1995, 1997; Oaksford & Chater, 2007; Politzer & Bourmaud, 2002). More recently, a different approach has been adopted that follows de Finetti's (1937) coherence-based probabilty theory (Coletti & Scozzafava, 2002; Gilio, 2002; Pfeifer & Kleiter, 2006, 2009 for a theoretical exposition, and Pfeifer & Kleiter, 2010, 2011 for psychological studies). It is this latter approach that will be taken, to which the tank analogy will be applied. We will be concerned with sentences that express unconditional or conditional events, the probability of which is represented by the components of the tank (compartments and volumes of liquid).

Boolean operations on elementary sentences can be represented in the tank analogy and these will be used when applicable to represent the premises and conclusions of the inference schemas.

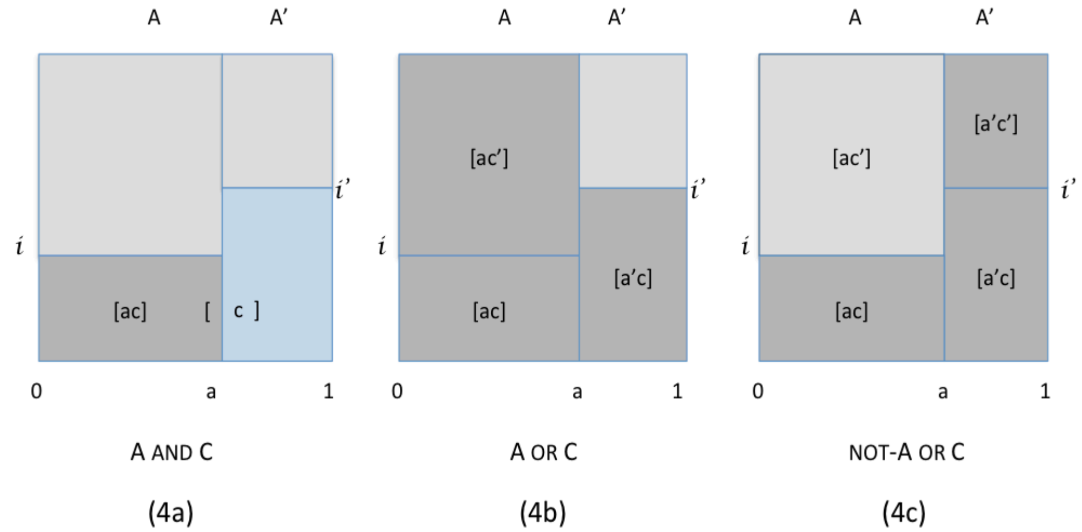


Figure 4. Boolean operations.

The volume of liquid contained in A (the dark shaded area [ac] in Figure 4a) is, *eo ipso,* the volume common to the compartment A and to C, so that it represents the conjunction A AND C. Disjunction A OR C is represented by the volume occupied by A or by the liquid (the dark shaded area in Figure 4b). Similarly, the material conditional (NOT-A OR C) is represented by the volume that is occupied by A' or by the liquid (Figure 4c).

In principle, any number of atomic sentences (and the associated events) can be represented because adding a sentence only requires further division of each compartment vertically[3] into two parts, and so on recursively. Most inference schemas discussed in the literature involve two or three atomic sentences, which will be the case in what follows. As is the case with mosaic displays, the limitation in the number of variables is only practical.

The fundamental question that we are going to address is "What constraints do the probability of the premises impose on the probability of the conclusion?" A major choice for probabilistic logics is to define a notion of validity that is the counterpart of logical validity in standard deductive logic. Within the present analogy, the mark of validity will be the physical feasibility of finding values for the tank's constituents specified by the premise and the conclusion at the same time. In addition, we will consider a few properties of deductive inference schemas under uncertainty which, by defining different types of probability preservation, represent different possible notions of validity. These properties are orthogonal to the tank analogy and they are proposed only for their possible psychological relevance. A given schema does or does not possess each of these properties.

In general, it seems desirable and possibly critical for reasoners to assess whether, or to be aware that:

1) Their degree of belief in the conclusion is higher, equal, or lower than it is in the premises. When the probability of the conclusion is no lower than the probability of at least one of the premises we will call the inference "conservative", and "dissipative" when it is no higher. More formally, an inference is conservative just in case P(conclusion) ≥ min P(premises), and it is dissipative just in case P(conclusion) ≤ max P(premises). These properties are not the negation of each other. An inference can be neither conservative nor dissipative.

2) Their degree of belief in the conclusion is high for some values of their degree of belief in the premises. In that case we will call the inference "forceful".

3) Highly believable premises warrant a highly believable conclusion. In that case we will say that the inference is "robust". This entails forcefulness, but not reciprocally, and is entailed by conservativeness, but not reciprocally[4].

We will consider below, on a case by case basis, some refinements on these broadly defined properties. In particular, what is meant by a "high" degree of belief will be specified (see section 3.1, 3.10, and 3.11).

In sum, inference schemas will be viewed as a relation between premises and conclusion characterizable by the limits in the degree of belief that it is rational to assign to the conclusion as a function of the degree of belief in each premise. (In the extreme case where the probability of the conclusion lies in the whole interval [0, 1], it is totally uninformative and the inference is useless).

The general method that we will follow consists in interpreting the premises and the conclusion as events and/or conditional events and translate them in terms of partitions of the tank and filling of the parts so defined, with volumes for events and levels for conditional events representing the values of the probabilities. Then we examine the probability constraints imposed by the premises on the conclusion.

3.1. And-introduction: A; C ∴ A AND C

We wish to determine the probability of the conjunction A AND C knowing the probability of its components, A, and C. That is, given the capacity a and the amount c to be poured into the tank, what can be the part of C that is in A? (see Figure 2). Trivially the part of C that could be in A, [ac], cannot exceed a nor can it exceed c itself: [ac] ≤ min {a, c}. Does the content of A have a lower bound? Filling A' as much as we can, if c is smaller than its capacity (c ≤ 1 - a), A remains empty and [ac] = 0; if c > 1 - a, A receives c - (1 - a), so that: [ac] ≥ max {a+c-1, 0 }. Hence:

P(a) = a; P(C) = c; P(A AND C) ∈ [max {0, a+c-1}, min {a, c}]

Breaking the upper bound amounts to committing the *conjunction fallacy* by which the conjunction of two events is estimated as more likely than one of the conjuncts. In the present case it amounts to either trying to pour more liquid into A than its capacity allows, or claiming that the liquid that is in A exceeds the whole volume of liquid.

This inference is **dissipative**, as the upper bound equals the minimum of either premise. It is also **robust**, as the lower bound is close to 1 when both premises have a probability close to 1. Take for instance a value of .90 to conventionally represent a "high" probability for the conclusion. This probability is warranted if the mean of the premises' probabilities is ≥ .95.

3.2. If-introduction: A; C ∴ IF A THEN C

We wish to determine the probability of the conditional *if A then C* knowing the probability of its components, A and C. The question amounts to the following: Given a (the capacity of A), and the amount c to be poured into the tank, what are 1) the highest, and 2) the lowest possible levels in A?

1) One fills up A first (see Figure 5a). Trivially, i = 1 if c ≥ a (A is filled up when there is as much or more liquid than A can receive), and i = a/c otherwise, hence: i ≤ min {c/a, 1}.

2) One fills up A' first. If c ≤ a', A remains empty and i = 0. If c > a', A' is filled up and the excess is poured into A (see Figure 5b) which receives c - a', hence its level (c - a') /a, that is, (c -1+a) / a. Hence : i ≥ max {0, (c -1+a) / a }. Therefore:

P(a) = a; P(C) = c; P(C/A) ∈ [max {0, (c -1+a) / a }, min {c/a, 1}]

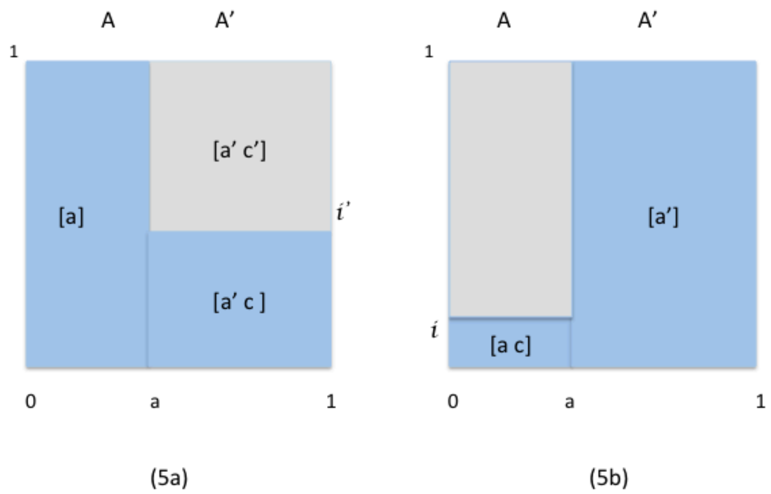


Figure 5. IF-introduction.

This inference is neither conservative nor dissipative, as the bounds can reach 0 while a and c differ from 0, or reach 1 while a and c differ from 1. Whenever a and c are high, so is i, and the inference is **robust**.

3.3. And-elimination: A AND C ∴ A A AND C ∴ C

The amount of liquid in A, [ac], being fixed, it is trivially smaller than, or equal to, either the capacity of A, a, or the total amount of liquid c, that is: [ac] ≤ a, and [ac] ≤ c. Or equivalently, if A contains [ac], its capacity cannot be less than [ac] (but it can contain more); and similarly if some of the liquid is in A there cannot be less liquid overall but there can be some more in A'. Therefore this inference is **conservative.**

P(A), P(C) ∈ [P(A AND C), 1]

3.4. 'And' to 'if': A AND C ∴ IF A THEN C

What are the lowest and highest levels in A, knowing the content of the left compartment? The lowest level is obtained when A has the greatest capacity, that is when A has the greatest base, a = 1 (Figure 6a). Now moving the partition leftwards (disregarding the content of A', c being free to vary), i can only increase (Figure 6b), showing that we always have i ≥ [ac]. The limit i = 1 is reached when [ac] = a (Figure 6c). Arithmetically of course

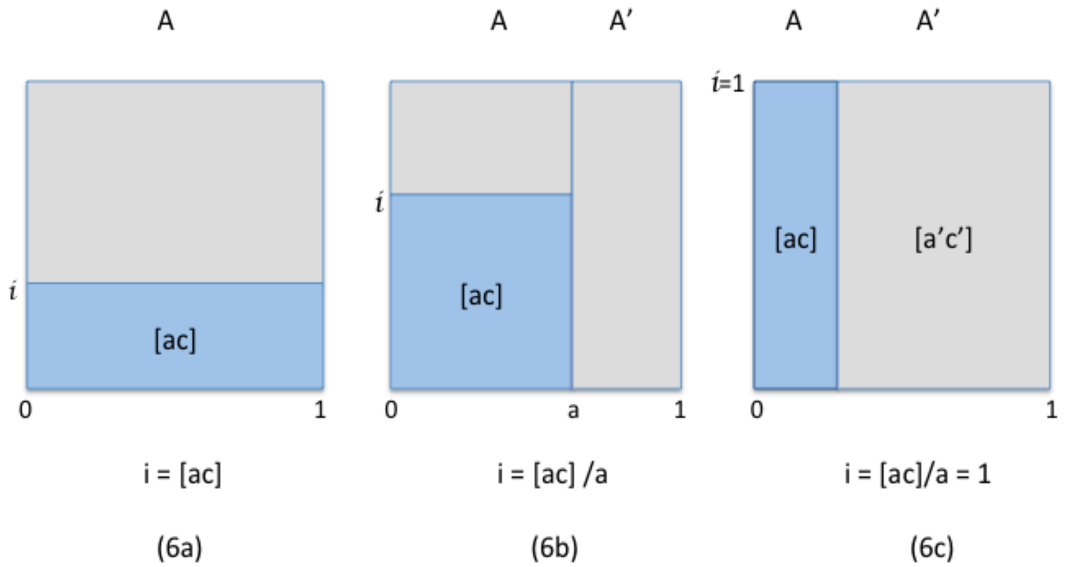


Figure 6. Inference from AND to IF.

i ≥ [ac] because the ratio i = [ac] /a is ≥ 1). In other words, if an amount of liquid in A at least as great as [ac] is warranted, then a level i that is at least as high as [ac] is also warranted, meaning that the inference from *and* to *if,* is **conservative.**

$$\mathbf{P(C/A) \in [\, P(A\ \text{AND}\ C), 1\,]}$$

3.5. Proof by cases: IF A THEN C; IF NOT-A THEN C ∴ C

The levels of liquid i in A and i' in A' being fixed, we are looking for the volume c (see Figure 7a). We consider what happens when a varies from 0 to 1. If i = i', there is independence and c is determined and equal to i and i'. If not, assume i > i'. When a reaches 0, the level in the tank equals the volume in A', c = i' and cannot be smaller (Figure 7b). When a reaches 1 the level in the tank equals the volume in A, c = i and cannot be greater

(Figure 7c). By continuity c varies between i' and i when a varies between 0 and 1 (Figure 7a), hence a **conservative** inference:

$$\mathbf{P(C/A) = i; \quad P(C/not\text{-}A) = i'; \qquad P(C) \in [\ \min \{i, i'\}\ ,\ \{\max \{i, i'\}\ ]}$$

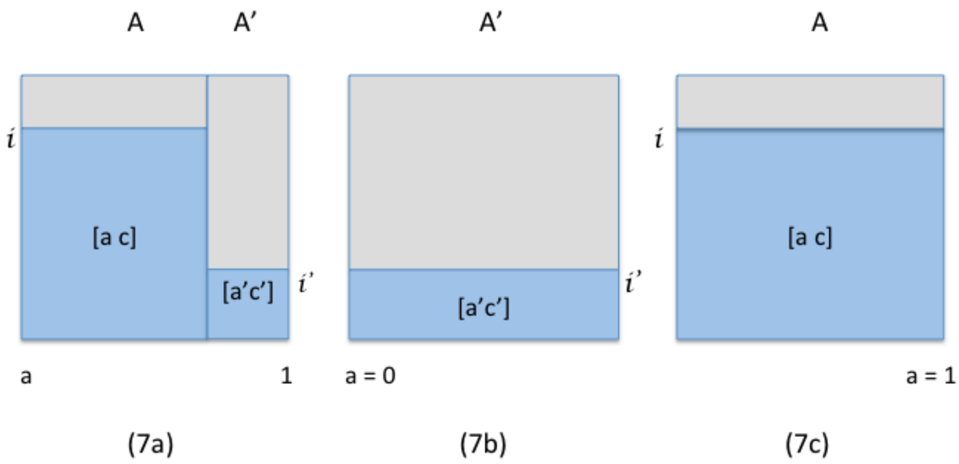


Figure 7. Proof by cases.

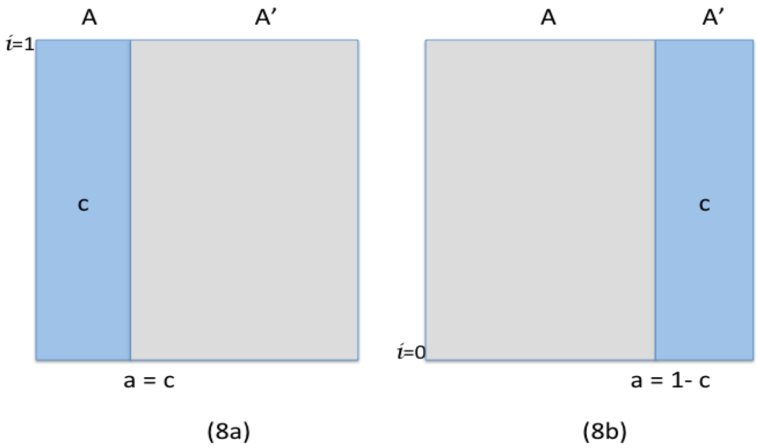
3.6. 'Not-A' to 'if': NOT-A ∴ IF A THEN C

So long as the overall content is unknown, knowing the capacity of the right compartment does not give information about the level of the liquid in the left compartment; therefore the latter can have any value and so does the confidence in the conclusion:

$$\mathbf{P(NOT\text{-}A) = a'; \qquad P(C/A) \in [0, 1]}$$

3.7. Consequent to 'if' : C ∴ IF A THEN C

What is the probability of the conditional, knowing the probability of its consequent? In other words, what is the level in the left compartment A, knowing the amount of liquid c in the tank ? For i to reach 1 (A full), it will suffice that there is enough liquid to fill it up, that is, the capacity of A should be no greater than the volume of liquid, which is always possible to fix (see the limiting case where a = c, Figure 8a). Similarly for i to be null (A empty) it suffices for the volume of liquid it contains to be smaller than the empty volume in A', which is automatically satisfied when c is small, and possible to fix when c is high by reducing and limiting A under a value such that a ≤ 1- c (Figure 8b). By continuity any intermediate level for i can be obtained when a varies between the two limiting positions. Therefore any confidence value can be given for the conclusion:



P(C) = c; P(C/A) ∈ [0, 1]

Figure 8. Inference from Consequent to IF.

3.8. 'If' to 'not-A or C' ('if' to the material conditional): IF A THEN C ∴ NOT-A OR C

Knowing the level i in the left compartment A, what is the volume occupied by the whole right compartment A' or by the liquid? Its measure is never lower than the measure of i because, as shown in Figure 9, there is always some space in A' above the level i up to the top (the darker area whose measure equals (1- a)(1- i)). P(not-A or C) is measured by (1x i) + (1- a)(1- i) = i + (1- a)(1- i). The difference between the probabilities of the premise and the conclusion is (1- a)(1- i) which is positive or null. It is maximal when i = 0 and null when i = 1. This inference is **conservative**: the probability of the conditional never exceeds that of the material conditional:

P(NOT-A OR C) ∈ [P(C/A), 1]

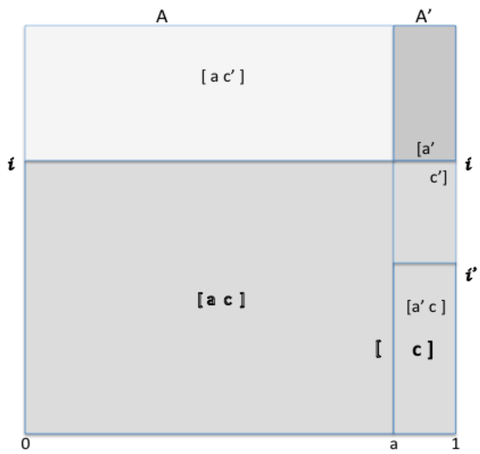


Figure 9. Inference from IF to NOT-A OR C.

3.9. Or-introduction: A; C ∴ A OR C

Knowning the capacity of A, and given a volume c of liquid, what are the smallest and the largest possible volumes that their union occupies? If we pour the liquid into A, either it can be contained in A (when $c \le a$, Figure 10a) making the total volume equal to a; or it cannot (meaning $c > a$, Figure 10b) and makes the total volume equal to c, hence the lower bound: max {a, c}. To get the largest possible volume we start pouring the liquid into A' and stop when the liquid is exhausted, which may occur before (or when) reaching the top of A' ($a+c \le 1$, Figure 10c) yielding a volume equal to a+c, or after filling up A' ($a+c > 1$, Figure 10d) , hence the upper bound: min {a+c, 1}. This inference is **conservative**.

$$P(A) = a; \quad P(C) = c; \qquad P(A \text{ OR } C) \in [\ \max\{a, c\}, \min\{a+c, 1\}\ ]$$

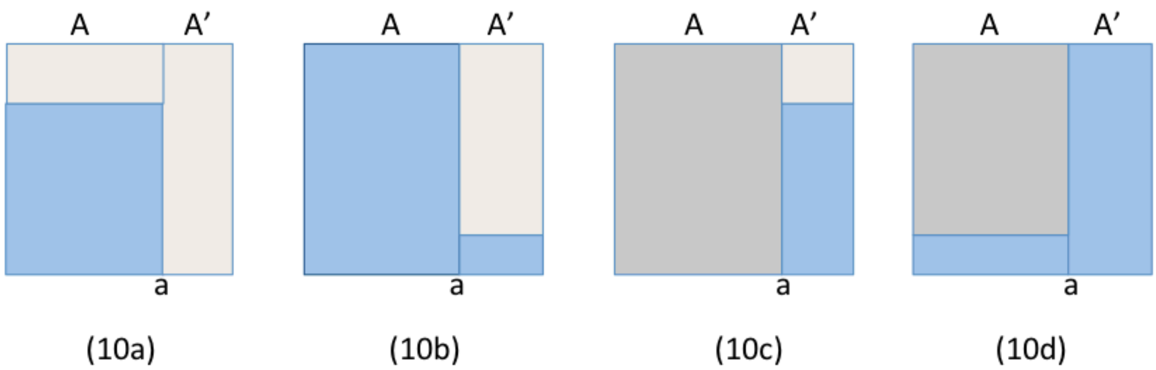


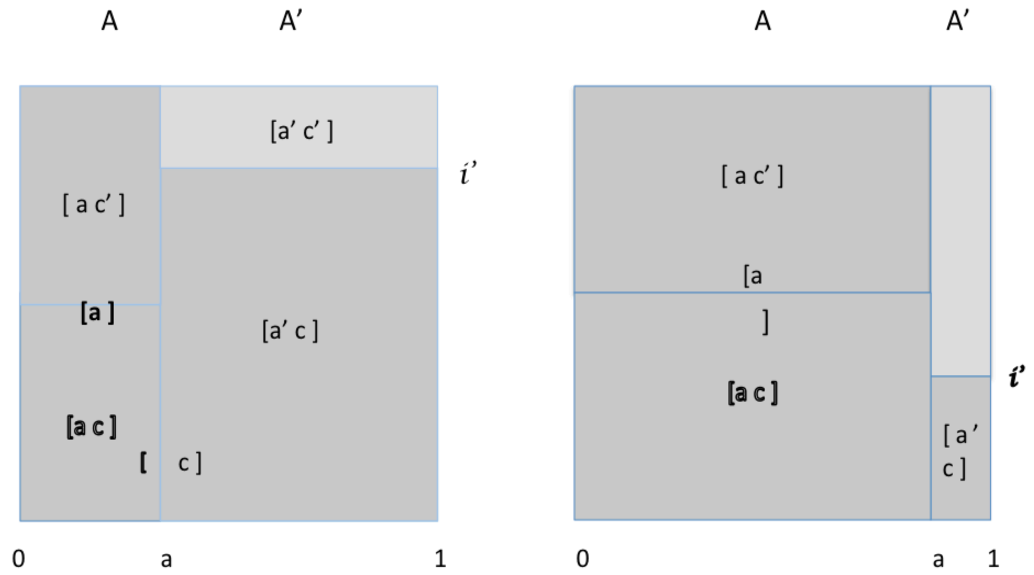
Figure 10. OR-introduction.

3.10. 'Or' to 'if not-': A OR C ∴ IF NOT-A THEN C

The conclusion represents the level in the right compartment, while the premise represents the volume occupied by the left compartment or by the liquid, that is, the capacity of A augmented with the content of A' (Figure 11a). Obviously, the empty space [a'c'] in A' is greater (or equal) relatively to A' than it is relatively to the whole tank; taking the respective complementary ratios, it follows that i' is smaller than the disjonction (or equal, in the limiting case where a=0). The probability of the conclusion i' can reach but never exceed that of the premise A OR C: the inference is **dissipative**. To find the lower bound, we notice (Figure

11b) that when a is large enough the contribution of the content of A' to A OR C is negligeable, that is, i' can have any value, in particular it can be small and even null. Hence:

(IF NOT-A THEN C) ∈ [0, P(A OR C)]



(11a) (11b)

Figure 11. Inference from OR to IF-NOT.

Several authors have discussed how "reasonable" (Stalnaker, 1975) or "plausible" (Edgington, 1986) it is to make this inference, or how "strong" it is (Gilio & Over, 2012) and have offered a justification for it. In the present framework, we have seen that a high probability for the premise does not warrant a high probability for the conclusion; in other words the inference is not robust. So, we ask whether there are special circumstances in which a high probability for the conclusion could be warranted. We already know that the inference is dissipative, so we can only hope for a condition, if any, under which the loss in confidence will not be too great.

As already noticed, when A is large, i' can be low or even null. In this case it is possible for *A or C* to be highly believable while *if not-A then C* is not, there is a great loss in confidence and this is why the inference is not robust. To avoid this situation, it suffices that A be not too large (Figure 11a). When this is the case, a high value for the disjunction entails a relatively high level in A', that is, i' is relatively high, and so the loss in confidence from premise to conclusion is limited. In sum, when there is high belief in *A or C*, to warrant a close degree of belief in *if not A then C*, we have to make the assumption that A is not too large. This qualitative result can be made precise as follows. Denoting the probability of the premise A OR C by d, and since i' = [a'c] /a', we get:

$$\mathbf{i' = (d - a) / (1 - a)}$$

(see Gilio & Over, 2012 for the same result by other means).

The graph of i' plotted against d with various values of a as a parameter is presented in Figure 12. It shows the linear increase of i' as a function of d. It illustrates that we always have i' ≤ d, and that the inference is (non strictly) dissipative (the exception occurs trivially when a = 0 and then i' = d = c) and that the limits for i' are 0 ≤ i' ≤ d.

When d is large, there is a possibility that i' be low (or even null); this occurs only when a is large. The vertical lines show that for a given value of d (in particular when d is large as has been assumed) i' is all the greater as a decreases. In brief, for modarate values of a the inference behaves like a robust inference. Take for instance d=.95, then i'=.90 for a=.50; but for a=.95 i' collapses to zero[5].

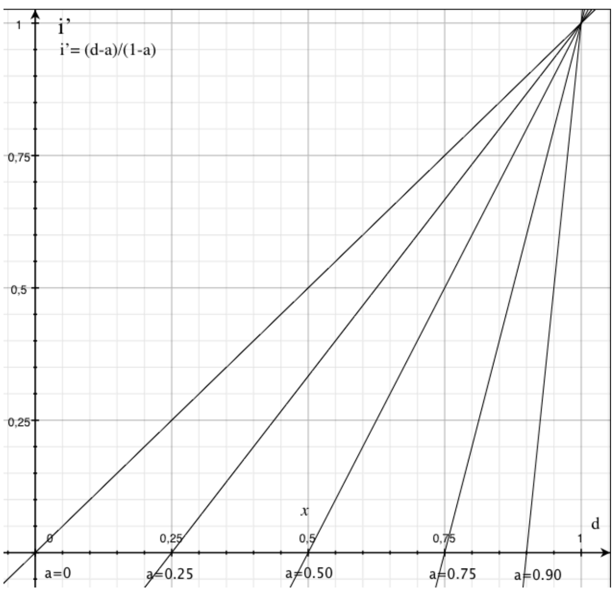


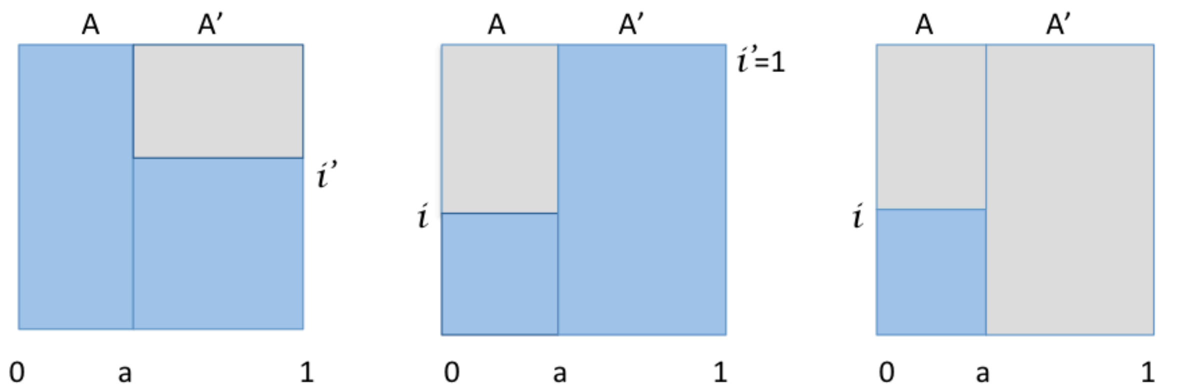
Figure 12. Inference from OR to IF-NOT. Variation of i' = P(C/NOT-A) as a function of d = P(A OR C) for various values of a = P(A).

This analysis concurs with Gilio & Over (2012) who discuss in depth the role of the magnitude of P(A) relative to P(A or C) (their measure of constructivity) in defining the weak versus the strong inference[6].

3.11. Contraposition: IF A THEN C ∴ IF NOT-C THEN NOT-A

The conclusion represents the degree to which the right compartment contributes to the vacuum (the emptiness of the tank). This degree will be denoted by i_c.

Consider the two extreme cases. One, when the left compartment is full (i = 1, full belief in the premise) all the empty volume is in A' so that i_c = 1 and the contrapositive is fully believable (see Figure 13a).



(13a) (13b) (13c)

Figure 13. Contraposition.

Two, when A occupies the whole tank (a = 1), the empty volume is entirely in A, and i_c = 0. In the general case the contribution of the right compartment varies from zero when it is full (i_c = 0, see Figure 13b) to a maximum when it is empty, yielding i_c = (1 - a) / (1 - c) (Figure 13c). But this maximum is susceptible of variation depending on the relative size of the compartments: as A decreases A' increases and the contribution of A' increases until i_c

reaches 1. In sum, The contrapositive can have all the degrees of confidence between 0 and 1.

P(C/A) = i; P(NOT-C/NOT-A) $\in$ [0, 1]

The foregoing result has been obtained under the assumption that i is the only known variable. However, it is interesting to study the constraints superimposed by other variables (a and i') in order to know under which circumstances the contrapositive has different bounds. Strictly speaking, in doing so, we will be considering a new inference defined by the addition of two premises (a and i'). For this purpose it suffices to express i_c as the ratio of the empty space in A', (1 - a)(1 - i') to the whole empty space 1-c:

$i_c = (1 - a)(1 - i') / (1 - c)$

$i_c = (1 - a)(1 - i') / [(1 - a)(1 - i') + a(1 - i)]$

This is the fundamental equation for contraposition expressing the functional relation between the contrapositive i_c and the conditional i, with a and i' as parameters. After the change of variable $y = i_c$ and x = i, it becomes:

$y = (1 - a)(1 - i') / [(1 - a)(1 - i') + a(1 - x)]$ (Eq)

We rewrite (Eq) as:

$y = u /(v - ax)$, with $u = (1- a)(1- i')$, and $v = u + a$.

It can be verified that this function is strictly increasing (first derivative strictly positive) and that it is concave (second derivative strictly positive).

Knowing whether the inference is conservative or not amounts to knowing the conditions under which y ≥ x, that is: $u /(v-ax) \geq x$, or:

$ax^2 - vx + u \geq 0$ (Ineq)

The associated equation can be factorised as:

$a(x-1)(x - u/a) = a(x-1)(x-j)$ where $j = u/a = (1- a)(1- i') /a$.

As a result, the sign of the difference y-x is determined as a function of a as follows:

• if j ≥ 1, there is no solution (other than x=1) within [0, 1], and (Ineq) is always satisfied, which means that for all i $\in$ [0, 1], $i_c \geq i$.

• if j < 1, there is a solution smaller than 1 which means that (Ineq) is satisfied in an interval [0, j] and not satisfied in [j, 0].

The condition j ≥ 1 is equivalent to i' ≤ (1-2a) / (1-a) or to a ≤ (1- i') / (2-i'), and similarly the condition j < 1 is equivalent to i' > (1-2a) / (1-a) or to a > (1- i') / (2-i'), so that in summary:

when a ≤ (1- i') / (2-i'), the inference is **conservative** ($i_c \geq i$) for all **i** $\in$ **[0, 1],**

when a > (1- i') / (2-i'), the inference is **conservative** ($i_c > i$) for **i** $\in$ **[0 , (1-a)(1-i') /a] ,**

and **dissipative** ($i_c < i$) for **i** $\in$ **[(1-a)(1-i') /a , 1].**

Figure 14 illustrates this discussion for the value of the parameter i' fixed arbitrarily at i' = .4 and for several values of the parameter a ranging from 0.2 to 0.95. It appears that the inference can be forceful only when i is high. When this is satisfied, inferring the contrapositive may be viewed as quite reasonable[7]. For instance, with a=0.5 and i'=0.4, for a probability of the conditional i=.95, the probability of the contrapositive i_C equals 0.92.

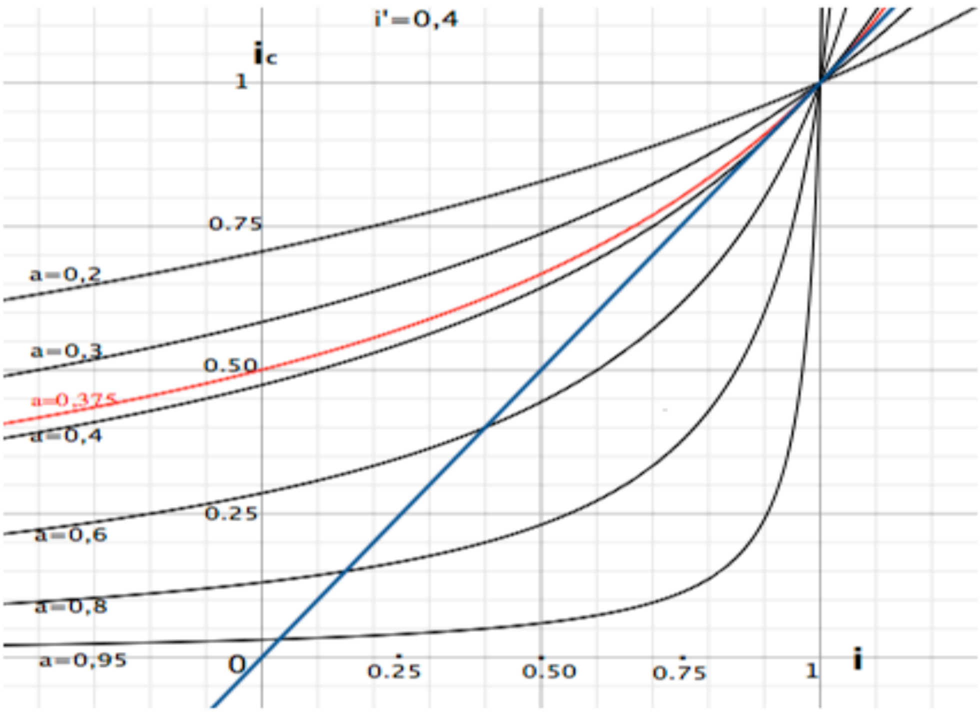


Figure 14. Probability of the contrapositive i_c plotted against i = P(C/A) for various values of a = P(A) and for i' = P(C/not A) fixed arbitrarily at 0.4. $i_c \geq i$ for 0 ≤ i ≤ j (j = intersect of f(i) and the diagonal).

3.12. Modus Ponens: IF A THEN C; A ∴ C

The combination of the premises a and i yields a x i = [ac] as the content of A. Obviously the content of the whole tank, c cannot be less, which gives a lower bound for the confidence in C: one must not be less confident in the conclusion than in the product of the levels of confidence in the premises. On the other hand, A being limited to level i, the tank cannot be full; c cannot exceed the content of A, a x i, augmented with the capacity of A', that is, (a x i) + (1 - a) (see Figure 15). Therefore:

P(C/A) = i; P(A) = a; P(C) ∈ [a x i , (a x i) +1- a]

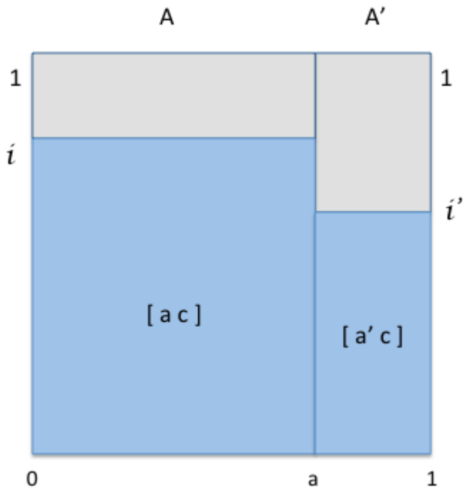


Figure 15. Modus Ponens.

Modus Ponens (MP) is not conservative, as its lower bound a x i is smaller or equal to the confidence in either premise (a, i); it is not dissipative, as its upper bound is greater or equal to the confidence in one of the premises, i; but it is **robust**, as a high confidence in the premises (a, i) warrants a high confidence in the conclusion because c ≥ a x i.

Notice that when A occupies the whole tank, the level in the tank is given by i: c = i and so c is known with the least uncertainty (as equal to i). Now suppose A decreases while i is fixed (say = 1). A' increases and because i' is unknown the uncertainty on c increases; when a becomes null A' occupies the whole tank and c can be anything between 0 and 1: this is the non probabilistic case of Denying the Antecedent (DA) where i=1 and a=0. This shows that

MP and DA are the two sides of a same coin; the variable is A in one case and A' in the other case.

3.13. Denying the Antecedent IF A THEN C; NOT-A ∴ NOT-C

The same equation as for MP can be used, replacing c with c' =1- c, which yields:

$$P(\text{NOT-}C) \in [\ a(1- i)\ ,\ 1 - (a \times i)]$$

This represents the degree of emptiness of the tank knowing the level in A and the size of A'. The equation shows that it is rational to have a level of confidence in the conclusion of DA that lies between the bounds indicated. In particular, trivially, it would be irrational to assess the total void as lower than a - (a x i), that is, lower than the void in A.

The result can also be given in terms of a' = P(NOT-A):

$$P(C/A) = i;\quad P(\text{NOT-}A) = a' \qquad P(\text{NOT-}C) \in [(1- i)(1 - a')\ ,\ 1 - i\,(1 - a')]$$

3.14. Affirming the Consequent: IF A THEN C; C ∴ A

Given the level in A and the content c, trivially there cannot be more liquid in A than the whole amount c, that is: a x i ≤ c, hence: a ≤ c / i (which is automatically satisfied when i ≤ c since a ≤ 1). Similarly, A cannot incorporate more void than the whole void in the tank, that is: (1 - i) x a ≤ 1- c, hence a ≤ (1- c) / (1- i) (which is automatically satisfied when c ≤ i since a ≤ 1). It follows that:

a ∈ [0, c / i] if i ≥ c, and a ∈ [0, (1- c) / (1- i)] if i ≤ c, which can be summarized by:

$$P(C/A) = i;\ P(C) = c; \qquad P(A) \in [\ 0, \min \{\ c / i,\ (1- c) / (1- i)\ \}\ ]$$

Figure 16 represents the graph of a plotted against c for two values of i as a parameter.

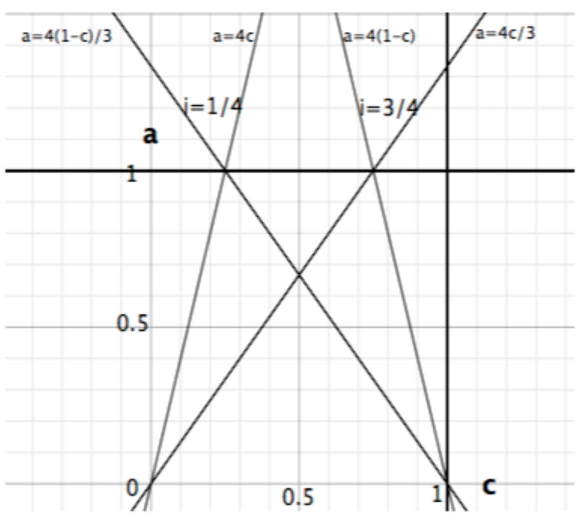


Figure 16. Inference of Affirming the Consequent. Graph of the conclusion a = P(A) as a function of the minor premise c = P(C) for two values of the major premise i = P(C/A).

For a given value of i, the range of a is obtained by referring to the triangle whose base is [0, 1] and the apex on the line a=1. For any value of c, the ordinate of the corresponding side of the triangle provides the range of a. It is noteworthy that when c nears 1 the upper limit of a decreases (and becomes null when c=1). If it is known that the left part is full to, say, three quarters but also that the tank is nearly full, the left part must be small. There is an exception to the decrease of a as c increases, which occurs when i=1 and c=1, in which case $a \in [0, 1]$, which is the non probabilistic fallacy of AC. This can be viewed as a special case of what happens when i and c are equal: the equality of the level in A with the degree of filling of the tank means independence, so that the position of the partition can be anywhere, in other words the inference is totally uninformative.

Suppose now that one has some belief in i (i > .5) and disbelieves c (c < .5): the range of a is measured by the ordinate on the left side of the triangle: the more c decreases, the more a decreases. This is the probabilistic form of Modus Tollens, showing its link with AC. In particular when i=1 a must be null: this is the non probabilistic MT. The solution of MT follows.

3.15. Modus Tollens: IF A THEN C; NOT-C ∴ NOT-A

Knowing the level in the compartment A and the size of the vacuum in the tank, what are the limits of the compartment A'? When A has a lower limit, A' has an upper limit, and vice-versa, so that we can use the result for the Affirmation of the Consequent after introducing a' = 1 - a and c' = 1 - c :

$a' \in [(i - 1 + c') / i , 1]$ if $i \geq 1 - c'$; and $a' \in [(1 - i - c') / (1 - i) , 1]$ if $i \leq 1 - c'$, which can be summarize by:

P(C/A) = i; P(NOT-C) = c'; P(NOT-A) $\in$ [max { (1 - i - c') / (1 - i), (i - 1 + c') / i } , 1]

The robustness of MT can be verified with a couple of values such as 0.92 for the premises i and c', which yields a lower bound of 0.91. A high level in the compartment A together with a large vacuum in the tank necessitates that the capacity of A be limited, failing which there would be a large amount of liquid in A and therefore in the tank.

3.16. Hypothetical syllogism: IF A THEN B; IF B THEN C ∴ IF A THEN C

The tank is initially filled with a liquid B (Figure 17a) and then some part of B is replaced by a liquid C. (We assume that the liquids are non miscible). In the certain, classic, case the compartment A is entirely filled with liquid B and then the whole of liquid B is replaced by liquid C, so that in the end A is filled with C and the inference is valid. In the probabilistic case, even if the level in A is high (i_{BA} = P(B/A) high) and a part of the volume B is replaced (respecting P(C/B)), the replacement of the liquid B could be done by removing it from A while little or no liquid B is removed from A', making the level of C in AB low or even null (Figure 17b). On the other hand, when liquid C is poured, some of it may also be introduced in B' and this amount introduced in AB' could occupy all this empty space so that P(C/A) need not be limited to P(B/A) and could reach 1 (the top of A). In brief, any confidence value can be given for the conclusion:

$$\mathbf{P(B/A) = i_{BA}\ ;\ \ P(C/B) = i_{CB}\ ;\qquad P(C/A) \in [\,0,\ 1]}$$

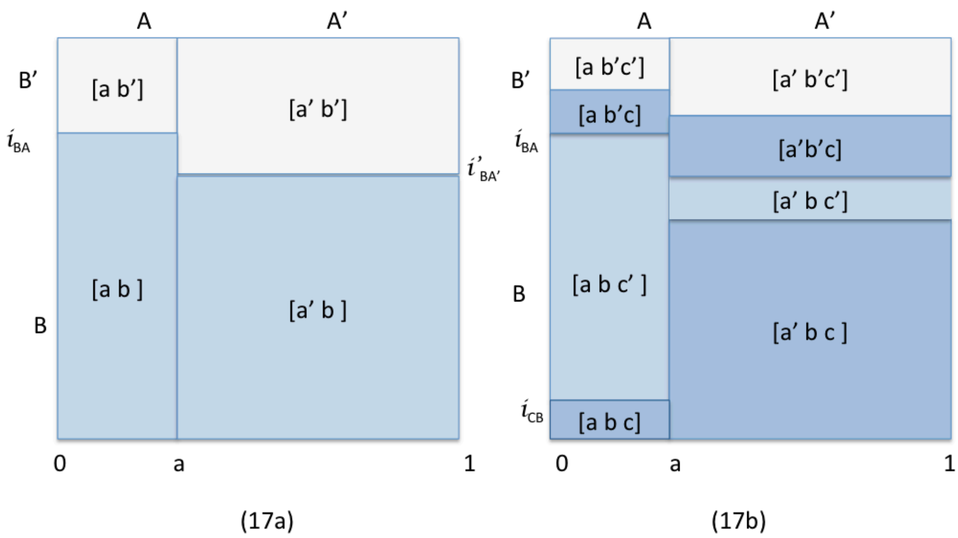


Figure 17. The hypothetical syllogism.

This is to be contrasted with the next inference, CUT.

3.17. CUT: IF A THEN B; IF A AND B, THEN C ∴ IF A THEN C

The difference with the hypothetical syllogism is that, as indicated by the second premise, when liquid C is introduced to replace B, this cannot be done without pouring at least some of it into A, so warranting that A will not be empty, that is, the probability of the conclusion cannot be null. The limits can be assessed as follows (Figure 17b):

Let i_{CB} stand for P(C/AB). Notice that the related level is relative to AB. As just explained, the amount in A cannot be less than the amount in AB, hence the minimum $i_{BA} \times i_{CB}$. Now, the maximum is obtained when in addition to the minimal amount the empty space AB' (=1- i_{BA}) is totally filled with liquid C, that is: $i_{BA} \times i_{CB} + (1- i_{BA})$, hence:

$$\mathbf{P(B/A) = i_{BA}; \quad P(C/AB) = i_{CB}; \qquad P(C/A) \in [\, i_{BA} \times i_{CB} \,,\, (i_{BA} \times i_{CB})+(1- i_{BA}) \,]}$$

This inference is not conservative, the lower bound being smaller than the confidence in either premise. It is not dissipative because the upper bound cannot be lower than the probability of one of the premises (i_{CB}). It is **robust** since the lower bound warrants a high value if both premises have a high probability.

3.18. Strengthening: IF A THEN C ∴ IF A AND B, THEN C

Compartment A is devided into two sub-compartments B and B'. In the sure case (Figure 18a) A is assumed to be full so that any subpart of it such as AB will be full as well and the conclusion follows with full confidence.

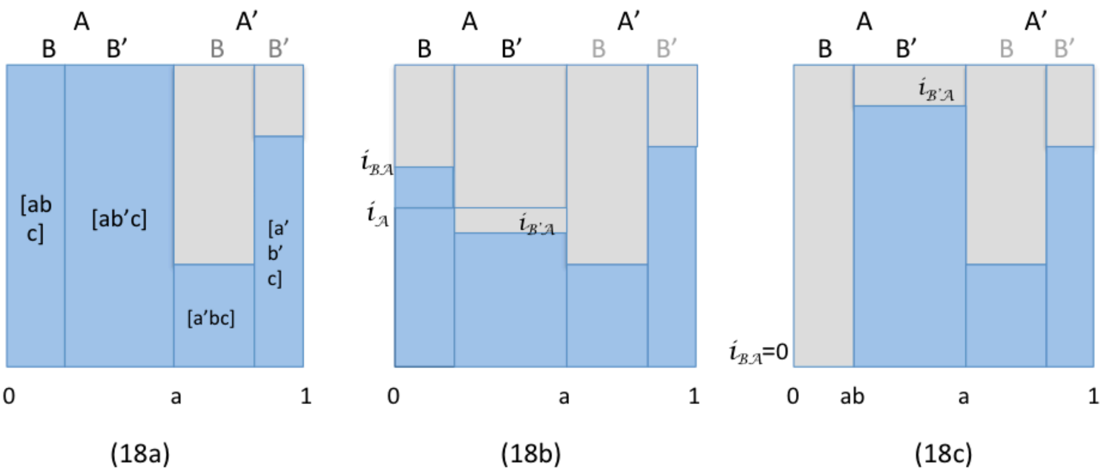


Figure 18. Strengthening of the antecedent.

In the probabilistic case there is no guarantee that A is full and the sub-compartments AB and AB' have levels i_{BA}= P(C/AB) and $i_{B'A}$ = P(C/AB'), respectively (Figure 18b). P(C/A) = i_A is a weighted value between i_{BA} and $i_{B'A}$. Strengthening the antecedent with B can increase or decrease the probability of the conditional P(C/A). In the case of Figure 18b, i_{BA} shows a higher probability than i_A.

To study the possible values of the conclusion it suffices to focus on the compartment A in which B and B' stand for new compartments and $\underline{c}$ is known; in other words, we conditionalize with respect to A, and the problem is now an inference already familiar (see section 3.7) from consequent C to 'if B then C' with B and B' as events within A. We know that no confidence value can be given, so that

$$\mathbf{P(C/A) = i; \quad P(C/\ (A\ \text{AND}\ B)) \in [0,\ 1]}$$

The extreme case where the level i_{BA} in AB equals 0 (Figure 18c) is of particular interest because, when dealing with causal conditionals, it coincides with a strict invalidating condition (or defeater). Assume a highly believable conditional *if A, C* asserted as a probable cause. As we have seen, for the level i_{BA} to be null when i_A is high, B must be small: this coincides with the well-known case where a conditional assertable with antecedent A must be retracted when A is qualified by an unexpected, or rare, or atypical defeater B.

The indeterminacy of the conclusion can be suppressed if the size of the AB compartment becomes a fixed parameter: adding *if A then B* as a premise yields the next schema, moving from monotonicity to cautious monotonicity.

3.19. Cautious monotonicity: IF A THEN C; IF A THEN B ∴ IF A AND B, THEN C

The level in A is $\underline{i}$, then A is partitioned into B and B'. The width of B in A is a x b (see Figures 18b and 18c). To know the level of liquid $\underline{i_{BA}}$ in AB we need to know how the liquid can be shared between AB and AB'.

1) AB contains an amount a x b x i; it can be emptied provided the empty space a(1- b)(1-i) is large enough to receive this extra liquid. In this case the minimum is 0. If not, there will remain in AB an amount equal to a x b x i - a(1- b)(1- i) = a(i + b-1), hence the level (i+b-1)/b.

2) AB can be filled up provided the amount of liquid in AB', i x a(1- b), is large enough. Otherwise the volume in AB equals i x a, hence the level i/b. In summary:

P(C/A) = i; P(B/A) = b P(C/ A AND B) ∈ [max {0, (i+b-1)/ b}, min {i/b, 1}]

With the addition of the premise *if A then B* the inference has become **robust**: we can have high confidence in *if A and B, then C* when, in addition to a high confidence in *if A then C* we have high confidence in *if A then B.* This inference is neither conservative nor dissipative, as the bounds can reach 0 while i and b differ from 0 or 1 while i and b differ from 1.

3.20. Right-nested conditionals and import-export:

IF A, THEN IF B THEN C ∴ IF A AND B, THEN C

A conditional is represented by a level and its consequent by a amount of liquid. In the premise of this inference the component "if B then C" is a level *qua* conditional but at the same time it must be an amount *qua* consequent of a conditional. At first sight, it would seem that this ambiguity prevents nested conditionals, and therefore the inference, to be represented in the system. However, a level is a relative amount and the level "C relatively to B" can be regarded as an amount taken in turn relatively to A, so that one may interpret the complex expression as a formulation of the conditional "if B then C" within the event A. That is, "if B then C" is conditioned on A and this allows the interpretation of the nested conditional. We start by representing the event A and then construct the partition (B, B'), each part of which accommodates its share of liquid. This procedural interpretation yields the same diagram as strengthening, and because it is reversible the inference is established as an equivalence. This equivalence can be translated in terms of conditional events. The premise is (C/B)/A (which is a well-formed expression in de Finetti's formalism), the conclusion is C/(A&B) and these denote the same conditional event (de Finetti, 1975, p. 328).

4. Conclusion

We have presented a device based on a physical implementation of probability theory (in the finite case) which allows the execution of deductions under uncertainty. At the basis of the derivation of the conclusions there are physical principles. We cannot get out of a compartment more than its actual content (no negative probability) nor can we pour an amount greater than its capacity (no probability greater than 1). It is also noteworthy that the physical constraint of conservation of the liquid (deeply supported by intuition) acts through operations such as (1) after filling up a compartment some extra liquid (if any) must be allocated to the other compartment, or (2) after moving the partition until the current level reaches the top requires that the extra liquid be accommodated in the other compartment. These operations, which have been used throughout, implement the total probability rule.

Note that the device can help answer the question of the probability of the conditional event when the probability of the conditioning event is null, or in another terminology, the probability of a conditional sentence whose antecedent's probability is null. Because it is hard to conceive of the level of a liquid in a container whose capacity is null, it would seem at first sight that it is undefined, as standard probability theory posits. However, consider the case in which a varies and i varies also between 0 and 1 (with c varying in accordance to allow this). Assume that a decreases and tends toward zero; then the level in A can continue to vary between 0 and 1 until a equals zero, so that $P(C/A) \in [0, 1]$ for $P(A) = 0$. Contrary to standard probability, the tank analogy need not define conditional probability as a primitive notion, so that the condition $P(A) \neq 0$ for the ratio P(A AND C) / P(A) to be defined is relaxed.

From a computational point of view, no claim to novelty of the *results* is made: indeed the probability bounds presented here can be obtained by de Finetti's (1937) method of linear equations (see also Coletti & Scozzafava, 2002; Pfeifer & Kleiter, 2009). The novelty with the analogy, besides the method, lies in the limited amount of computation and its simplicity. This is no wonder: once the components that represent the premises have been selected, the variation of the component of interest can be read off the diagram because the device incorporates the laws of probability. It is only when reading the bounds that the four elementary arithmetical operations are generally needed to express the value of the constituent of interest (the conclusion).

The tank analogy can be considered from two different points of view, practical and theoretical. Practically, one advantage linked to the operational procedure of the analogy over formal methods of calculation is that it provides both a basis for the quantitative calculation of the bounds and a qualitative method which opens didactic perspectives. Indeed, when the study of uncertain reasoning becomes sufficiently developed to reach wider audiences than specialized researchers or advanced students, the analogy may be a valuable tool to explain why, for instance, it is irrational to have too low or too high confidence in the conclusion of Modus Ponens, why contraposition is unwarranted (but which additional information/premise can warrant some confidence and how much), under which conditions Affirming the Consequent may not be a fallacy, and so on. Take for instance Modus Ponens. Individuals whose confidence in the conclusion is lower than the lower bound $P(A) \times P(C/A)$ commit themselves to the belief that the amont of liquid in the whole tank is less than the amount in one of its parts (the left compartment). Similarly, those whose confidence in the conclusion is higher than the upper bound $P(A) \times P(C/A) + 1 - P(A)$ commit themselves to the belief that there can be more liquid in the tank than its overall capacity.

Theoretically, the main interest of the analogy is to provide an interpretation of the meaning of probabilistic deductive inferences. Atomic sentences are interpreted in terms of volumes or capacity of compartments, and conditionals in terms of levels. Boolean expressions have their interpretation in terms of union or intersection of volumes or compartments so that the logical operations have their physical counterparts. Varying the value of the constituents (volumes, capacities and levels) provides the probability measures. Overall, an inference is viewed as a procedure which consists of singling out some of the constituents of the device (the premises) and examining whether their values constrain another specified constituent (the conclusion). The absence of constraint means that any value of the constituent is physically possible, the confidence in the associated sentence can be anywhere in the interval [0, 1] and so the inference is uninformative. The existence of a constraint means that some values of the constituent are excluded because they are physically impossible; this results in an interval of possible values that provide the limits in confidence in the conclusion and so the inference is informative.

In summary, we possess an operational procedure to evaluate normatively, that is, from the viewpoint of probability theory, deductive inferences under uncertainty. It can be seen that the evaluation of the value of the concluding constituent amounts to a search for the conditions of compatibility or coherence between this and the set of values permitted by the premises' constituents. It would be incoherent to accept the premises' constituents' values and expect a value for the concluding constituent that turns out to be physically impossible. This materializes de Finetti's notion of coherence and offers a counterpart of betting situations and Dutch book arguments used to assess individuals' rationality in their probability judgments: in a similar way that individuals whose bets do not conform to coherent probability bounds are doomed to losing their money, the same individuals are doomed to failure in attempting to execute fillings that are physically impossible.

Finally, that the coherence approach to the laws of probability can be materially implemented and simulated by physical principles may suggest some foundational reflection on probability theory.

Notes

1. The device can be described as a Cartesian product of two Bernouilli random variables $\Omega_1 \times \Omega_2$ with Ω_1 = {left, right} and Ω_2 = {(volume of) liquid, (volume of) empty space}.

2. Strictly speaking, the two events just denoted by A and C should receive different respective notations to distinguish them from the material constituents of the device, namely the compartment A and the liquid C. We keep a single notation so long as there is no risk of confusion.

3. The division may also be applied horizontally. This is a basic property of mosaic displays. Here the division may be applied horizontally to the filled and empty volumes of the compartments (see section 3.16).

4. Robustness and conservativeness are equivalent to Adams' (1996) High probability preservation and Minimum probability preservation, respectively.

5. By considering the values of a as a parameter, we have implicitly added a premise A, so that in this part of the discussion we have in fact considered the inference from *A or C; A* to *if not-A then C* and found that it is forceful.

6. Because there is asymmetry in the diagram between A and C, one may wonder whether the results remain identical after exchanging A and C in the conclusion (yielding an inference from A OR C to IF NOT-C THEN A). It can be verified easily that the answer is affirmative. That is, the inference is robust provided C is not too high. This shows again that the allocation of a sentence to one role, "horizontal" (the partition) or "vertical" (the liquid) is only a matter of visual presentation.

7. This discussion highlights that there is some ambiguity in referring to the contrapositive. The strict sense refers to the case where the conditional is considered in the absence of information about the belief in other events, in which case we have seen that the coherence interval is [0, 1]. In the wide sense, we still refer to the formally identical sentence *if not-C then not-A,* but now it is affected by the belief in other events (A and *if not-A then not-C*) and its probability interval changes accordingly.

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